ON SOME IDENTITIES AND SYMMETRIC FUNCTIONS FOR
LUCAS AND PELL NUMBERS

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Abstract. In this paper, we show how the action of the symmetrizing endomorphism operator $\delta_{e_1 e_2}$ to the series $\sum_{n=0}^{\infty} a_n e_1^a z^n$ allows us to obtain an alternative approach for the determination of Fibonacci and Lucas-Pell numbers.

1. Introduction and Notations

The second-order linear recurrence sequence $(U_n(a, b; p, q))_{n \geq 0}$, or briefly $(U_n)_{n \geq 0}$, is defined by

$$U_{n+2} = pU_{n+1} + qU_n, \quad U_0 = a, \quad U_1 = b.$$ 

Where $a$, $b$ and $p$, $q$ are arbitrary real numbers for $n > 0$. The Binet formula for the sequence $(U_n)_{n \geq 0}$ is

$$U_n = \frac{c_1 x_1^n - c_2 x_2^n}{x_1 - x_2},$$

where $c_1 = b - ax_2$ and $c_2 = b - ax_1$. Certain sequence of numbers that appeared here are Fibonacci number $(F_n)_{n \geq 0}$, if we take $p = q = b = 1, a = 2$, Lucas number $(L_n)_{n \geq 0}$ for $p = 2, q = b = 1, a = 0$, Pell number $(P_n)_{n \geq 0}$ and Pell-Lucas number $(Q_n)_{n \geq 0}$, when one has $p = b = a = 2, q = 1$. In this paper, we show that the use of the action of the symmetric endomorphism operator $\delta_{e_1 e_2}$ to the series $\sum_{n=0}^{\infty} a_n (e_1 z)^n$, gives an alternative approach for determining the generating functions of some sequences of numbers cited above.

Let $k$ and $n$ be two positive integer and $\{x_1, x_2, ..., x_n\}$ are set of given variables, recall [8] that the $k$-th elementary symmetric function $e_k(x_1, x_2, ..., x_n)$ and the $k$-th complete homogeneous symmetric function $h_k(x_1, x_2, ..., x_n)$ are defined respectively by

$$e_k(x_1, x_2, ..., x_n) = \sum_{i_1 + i_2 + ... + i_n = k} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}, \quad 0 \leq k \leq n.$$
with $i_1, i_2, \ldots, i_n = 0$ or $1$.

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{i_1 + i_2 + \cdots + i_n = k} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, \ldots, i_n \geq 0$.

First, we set $e_0(x_1, x_2, \ldots, x_n) = 1$ and $h_0(x_1, x_2, \ldots, x_n) = 1$ (by convention). For $k > n$ or $k < 0$, we set $e_k(x_1, x_2, \ldots, x_n) = 0$ and $h_k(x_1, x_2, \ldots, x_n) = 0$.

**Lemma 1** [10] The relations

1) $F_{-n} = (-1)^{n+1} F_n,$
2) $L_{-n} = (-1)^n L_n,$
3) $P_{-n} = (-1)^{n+1} P_n,$
4) $Q_{-n} = (-1)^n Q_n$

hold for all $n \geq 0$.

**Definition 1** Let $A$ and $E$ be any two alphabets, then we give $S_n(A - E)$ by the following form:

$$\prod_{x \in E} (1 - ez) \prod_{x \in A} (1 - az) = \sum_{n=0}^{\infty} S_n(A - E) z^n,$$

with the condition $S_n(A - E) = 0$ for $n < 0$ (see [1]).

**Corollary 1** Taking $A = 0$ in (1.1), that gives

$$\prod_{x \in E} (1 - ez) = \sum_{n=0}^{\infty} S_n(-E) z^n. \quad (2)$$

**Definition 2** [7] Given a function $g$ on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{x_i; x_{i+1}}(g) = \frac{g(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - g(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)}{x_i - x_{i+1}}.$$

It should be noted that the divided difference operator $\partial_{x_i; x_{i+1}}$ commutes with symmetric functions at $x_i, x_{i+1}$ and is compatible with the function $S_n$ [6].

**Definition 3** [2] The symmetrizing operator $\delta^{k}_{e_1 e_2}$ is defined by

$$\delta^{k}_{e_1 e_2}(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \text{ for all } k \in \mathbb{N}.$$

**Proposition 1** [3] Let $E = \{e_1, e_2\}$ an alphabet, we define the operator $\delta^{k}_{e_1 e_2}$ as follows:

$$\delta^{k}_{e_1 e_2} f (e_1) = S_{k-1}(e_1 + e_2) f (e_1) + e_2^k \partial_{e_1 e_2} f (e_1), \text{ for all } k \in \mathbb{N}.$$

2. The Main Result

In our main result, we will combine all these results in a unified way such that all these obtained results can be treated as special case of the following Theorem.
Theorem 1
Given an alphabet $E = \{e_1, e_2\}$ and two sequences $\sum_{n=0}^{+\infty} a_n z^n$, $\sum_{n=0}^{+\infty} b_n z^n$ such that $\left(\sum_{n=0}^{+\infty} a_n z^n\right) \left(\sum_{n=0}^{+\infty} b_n z^n\right) = 1$, then

$$\sum_{n=0}^{+\infty} a_n \delta_{e_1 e_2}^{k+n-1} (e_1) z^n = \frac{\sum_{n=0}^{k-1} b_n (e_1 e_2)^n \delta_{e_1 e_2}^{k-n} (e_1^{-1}) z^n - (e_1 e_2)^k \sum_{n=0}^{+\infty} b_{n+k+1} \delta_{e_1 e_2} (e_1^n) z^{n+1}}{\left(\sum_{n=0}^{+\infty} b_n e_1^n z^n\right) \left(\sum_{n=0}^{+\infty} b_n e_2^n z^n\right)}.$$ (3)

Proof. Let $\sum_{n=0}^{+\infty} a_n z^n$ and $\sum_{n=0}^{+\infty} b_n z^n$ be two sequences as $\sum_{n=0}^{+\infty} a_n z^n \times \sum_{n=0}^{+\infty} b_n z^n = 1$.

On one hand, since $f(e_1) = \sum_{n=0}^{+\infty} a_n e_1^n z^n$, we have

$$\delta_{e_1 e_2}^k f (e_1) = \delta_{e_1 e_2}^k \left(\sum_{n=0}^{+\infty} a_n e_1^n z^n\right) = \sum_{n=0}^{+\infty} a_n \delta_{e_1 e_2}^{k+n-1} (e_1) z^n,$$

which is the left hand side of (3). On the other hand, since

$$f(e_1) = \frac{1}{\sum_{n=0}^{+\infty} b_n e_1^n z^n},$$

we have that

$$\partial_{e_1 e_2} f(e_1) = \frac{1}{e_1 - e_2} \left(\sum_{n=0}^{+\infty} b_n e_1^n z^n - \sum_{n=0}^{+\infty} b_n e_2^n z^n\right) = \frac{1}{e_1 - e_2} \left(\sum_{n=0}^{+\infty} b_n e_1^n z^n - \sum_{n=0}^{+\infty} b_n e_2^n z^n\right) = \frac{\sum_{n=0}^{+\infty} b_n e_1^n z^n \sum_{n=0}^{+\infty} b_n e_2^n z^n}{\left(\sum_{n=0}^{+\infty} b_n e_1^n z^n\right) \left(\sum_{n=0}^{+\infty} b_n e_2^n z^n\right)}.$$
By Proposition 1, it follows that
\[
\delta_{e_1e_2}^k f(e_1) = S_{k-1}(e_1 + e_2)f(e_1) + e_2^k\partial_{e_1}^2 f(e_1)
\]
\[
= S_{k-1}(e_1 + e_2) - e_2^k \sum_{n=0}^\infty b_n S_{n-1}(e_1 + e_2) z^n
\]
\[
= \sum_{n=0}^\infty b_n \left[ e_2^k S_{k-1}(e_1 + e_2) - e_2^k S_{n-1}(e_1 + e_2) \right] z^n
\]
\[
\left( \sum_{n=0}^\infty b_n e_1^n z^n \right) \left( \sum_{n=0}^\infty b_n e_2^n z^n \right)
\]

Hence, we have that
\[
\delta_{e_1e_2}^k f(e_1) = \sum_{n=0}^{k-1} b_n (e_1 e_2)^k \delta_{e_1e_2}(e_1)^{-1} z^n - (e_1 e_2)^k \sum_{n=0}^\infty b_{n+k+1} \delta_{e_1e_2}(e_1)^n z^{n+1}
\]
\[
\left( \sum_{n=0}^\infty b_n e_1^n z^n \right) \left( \sum_{n=0}^\infty b_n e_2^n z^n \right)
\]

This completes the proof.

3. ON THE SYMMETRIC FUNCTIONS OF SOME NUMBERS

In this part, we derive the new generating functions of some known numbers. Indeed, we consider Theorem 1 in order to get Fibonacci numbers, Lucas numbers and Pell-Lucas numbers with \( k = 1 \) and \( k = 2 \), for the case \( \frac{1}{1+z} = \sum_{n=0}^\infty (-1)^n z^n \).

**Lemma 2** Given an alphabet \( E = \{e_1, e_2\} \), we have
\[
\sum_{n=0}^\infty (-1)^n h_n(e_1, e_2) z^n = \frac{1}{(1+e_1 z)(1+e_2 z)}, \quad \text{with} \quad h_n(e_1, e_2) = S_n(e_1 + e_2).
\]  

**Lemma 3** Given an alphabet \( E = \{e_1, e_2\} \), we have
\[
\sum_{n=0}^\infty (-1)^n h_{n+1}(e_1, e_2) z^n = \frac{e_1 + e_2 + e_1 e_2 z}{(1+e_1 z)(1+e_2 z)}, \quad \text{with} \quad h_{n+1}(e_1, e_2) = S_{n+1}(e_1 + e_2).
\]  

By replacing \( e_2 \) by \( -e_2 \) in (4) and (5), we obtain
\[
\sum_{n=0}^\infty (-1)^n S_n(e_1 + [-e_2]) z^n = \frac{1}{1+(e_1 - e_2) z - e_1 e_2 z^2},
\]  

\[
\sum_{n=0}^\infty (-1)^n S_{n+1}(e_1 + [-e_2]) z^n = \frac{e_1 - e_2 - e_1 e_2 z}{1+(e_1 - e_2) z - e_1 e_2 z^2}.
\]
Choosing $e_1$ and $e_2$ such that \[
\begin{align*}
e_1 e_2 &= 1 \\
e_1 - e_2 &= 1
\end{align*}
\] and substituting in (6) and (7) we get
\[
\sum_{n=0}^{\infty} F_n z^n = \frac{1}{z^2 - z - 1},
\]
which represent a generating function for Fibonacci numbers such that $F_{-n} = (-1)^{n+1} S_n (e_1 + [-e_2])$.
\[
\sum_{n=0}^{\infty} (-1)^n S_{n+1} (e_1 + [-e_2]) z^n = \frac{1 - z}{1 + z - z^2},
\]
which is given by Boussayoud et al [3].

**Corollary 2** For $n \in \mathbb{N}$, we have
\[
S_{n+2} (e_1 + [-e_2]) = S_{n+1} (e_1 + [-e_2]) + S_n (e_1 + [-e_2]).
\]
Choosing $e_1$ and $e_2$ such that \[
\begin{align*}
e_1 e_2 &= 1 \\
e_1 - e_2 &= 2
\end{align*}
\] and substituting in (6) and (7), where we have
\[
\sum_{n=0}^{\infty} (-1)^n S_n (e_1 + [-e_2]) z^n = \frac{1}{1 + 2z - z^2},
\]
which yields also new generating functions.
\[
\sum_{n=0}^{\infty} (-1)^n S_{n+1} (e_1 + [-e_2]) z^n = \frac{2 - z}{1 + 2z - z^2},
\]
Multiplying the equation (8) by 3 and subtract it from (9) we get
\[
\sum_{n=0}^{\infty} L_n z^n = \frac{2 + z}{1 + z - z^2},
\]
which represents a new generating function for Lucas Numbers.

**Corollary 3** For all $n \in \mathbb{N}$, we have
\[
L_{-n} = (-1)^n \left[ 3 S_{n+1} (e_1 + [-e_2]) - S_{n+1} (e_1 + [-e_2]) \right].
\]
Multiplying the equation (10) by (-2) and added to (11) we obtain
\[
\sum_{n=0}^{\infty} P_n z^n = \frac{z}{1 + 2z - z^2},
\]
which represents a new generating function for Pell Numbers.

**Corollary 4** For all $n \in \mathbb{N}$, we have
\[
P_{-n} = (-1)^{n+1} \left[ S_{n+1} (e_1 + [-e_2]) - 2 S_n (e_1 + [-e_2]) \right].
\]
Multiplying the equation (10) by 6 and added to (11) by (-2), we have
\[
\sum_{n=0}^{\infty} Q_n z^n = \frac{2 + 2z}{1 + 2z - z^2},
\]
which represents a new generating function for Pell-Lucas Numbers.

**Corollary 5** For all $n \in \mathbb{N}$, we have
\[
P_{-n} = (-1)^n \left[ 6 S_n (e_1 + [-e_2]) - 2 S_{n+1} (e_1 + [-e_2]) \right].
\]

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References


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