A NEW SEQUENCE REALIZING LUCAS NUMBERS, AND THE LUCAS BOUND

E. ÖZKAN, A. GEÇER, AND İ. ALTUN

Abstract. We define a set $L(n)$ of vectors with positive integral entries. We show that the cardinality of $L(n)$ is the $n$th Lucas number $L_n$, for $n \geq 1$. We then show that the number $l(n)$ of $M$-sequences of length $n$ is bounded by the Lucas number $L_n$, for $n \geq 1$. This is an analogue of similar statements with Fibonacci numbers $F_n$ instead of Lucas numbers.

1. Introduction

First, we recall the well-known definitions of the Fibonacci and Lucas numbers. The Fibonacci number $F_n$ is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, n \geq 2,$$

with seed values $F_0 = 0$ and $F_1 = 1$.

The Lucas number $L_n$ is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, n \geq 2,$$

with seed values $L_0 = 2$ and $L_1 = 1$.

In recent years, there has been much interest in applications of Fibonacci and Lucas numbers. They appear in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc. Also, these numbers have many important applications to diverse fields such as computer science, physics, biology, and statistics. We can see applications of the Fibonacci sequence in group theory in [4,6] and also see some generalized Fibonacci and Lucas sequences in [3,5].

In this work, we give an application of Lucas numbers and $M$-sequences. We recall some definitions.

A multiset is a finite sequence of elements that may contain repeated elements. The order of the elements is irrelevant, but their multiplicities are part of the structure. So, a multiset $S$ with elements in $\mathbb{N}$ can be uniquely written in the form $S = \{b_1, b_2, \ldots, b_{k-1}, b_k\}_{\leq}$, where $b_i \in \mathbb{N}$, $k \geq 0$, and the subscript "$\leq$" indicates that we have arranged the elements in weakly increasing order: $b_1 \leq b_2 \leq \ldots \leq b_k$. The dimension of a multiset $S$ is the number of elements in $S$, counting
multiplicities. So the multiset \( S \) above has dimension \( |S| = k \), and we call it a \( k \)-multiset. If \( k = 0 \) then \( S \) is empty and \( |S| = 0 \).

A multicomplex is a finite collection of multisets that is closed under taking submultisets. The dimension of a multicomplex is the greatest dimension of a multiset it contains. The \( m \)-vector of a multicomplex is \((m_0, m_1, m_2, \ldots, m_d)\), where \( m_i \) is the number of multisets of dimension \( i \) in the multicomplex. We have \( m_0 = 1 \).

Let \( M \) be a collection of monomials \( x_1^{i_1}x_2^{i_2} \ldots x_n^{i_n}, i_j \geq 0 \), such that if \( a \) is a monomial in \( M \), and \( b \) is a monomial which divides \( a \), then \( b \) is in \( M \). Then \( M \) is a multicomplex where the multisets are encoded as monomials. The dimension of the multiset is the degree of the monomial it defines.

A sequence is called an \( M \)-sequence if it is the \( m \)-vector of the multicomplex \( M \), where \( m_i \) is the number of monomials of degree \( i \) in \( M \). Rather than bounding the degree and number of variables, we fix the length \( \lambda(m) := \sum m_i \). This is the number of the elements in the multicomplex \( M \).

Following [2], for \( n \geq 1 \), let \( l(n) \) denote the number of \( M \)-sequences of length \( n \). In fact, the authors of [2] define the set \( M(n) = \{ m = (m_0, m_1, \ldots) \mid m \text{ is an } M \text{-sequence and } \lambda(m) = n \} \), and set \( l(n) = |M(n)| \) for its cardinality. The sets \( M(1), M(2), \ldots \) are mutually disjoint.

Using Macaulay’s Theorem, the authors of [1, Theorem 4.2.10] constructed all possible \( M \)-sequences of a given length. In Table 1, we list the \( M \)-sequences of length at most 7, from which we deduce that the first few terms of the sequence \( \{ l(n) \}_{n \geq 1} \) of [2] are the numbers 1, 1, 2, 3, 5, 8, 12. They are bounded by, but not equal to, the Fibonacci numbers.

<table>
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<td>2</td>
<td>3</td>
<td>5</td>
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</table>

**Table 1.** The \( M \)-sequences of length at most 7 obtained by using Definition 2.1. We write \( t_0t_1t_2 \ldots t_s \) for the \( M \)-sequence \((t_0, t_1, \ldots, t_s)\). E.g., 1221 is the \( M \)-sequence \((1, 2, 2, 1)\). For example, when \( \lambda = 4 \), the \( m \)-vector of \( M = \{ x_1^3, x_1^2, x_1, 1 \} \) is \((1, 1, 1, 1)\), of \( M = \{ x_1^2, x_1, 2 \} \) is \((1, 2, 1)\), of \( M = \{ x_1, x_2, x_3, 1 \} \) is \((1, 3, 1)\).

Let us recall the Fibonacci bound of [2] before we define the Lucas bound.
2. Fibonacci Bound

**Definition 2.1.** For \( n \geq 1 \), define recursively the Fibonacci set \( F(n) \) of vectors with entries in \( \mathbb{N} \) as follows: \( F(1) = \{(1)\}; \ F(2) = \{(1, 1)\} \);

For \( n \geq 3 \) define \( F(n) := C(n) \cup D(n) \) where (see [2])

\[
C(n) := \{(1, t_1, \ldots, t_s, 1) \mid (1, t_1, \ldots, t_s) \in F(n - 1)\},
\]

\[
D(n) := \{(1, t_1, \ldots, t_{s-1}, t_s + 1) \mid (1, t_1, \ldots, t_s) \in F(n - 1), \text{ with } t_{s-1} > 1 \text{ or } s = 1\}.
\]

The sets \( C(n), D(n) \) are disjoint for each \( n \). The sets \( F(1), F(2), \ldots \), are mutually disjoint.

For example,

\[
F(3) = \{(1, 1, 1), (1, 2)\};
\]

\[
F(4) = \{(1, 1, 1, 1), (1, 2, 1), (1, 3)\};
\]

\[
F(5) = \{(1, 1, 1, 1, 1), (1, 2, 1, 1), (1, 3, 1), (1, 2, 2), (1, 4)\};
\]

\[
F(6) = \{(1, 1, 1, 1, 1, 1), (1, 2, 1, 1, 1), (1, 3, 1, 1), (1, 2, 2, 1)(1, 4, 1), (1, 3, 2), (1, 2, 3), (1, 5)\};
\]

\[
F(7) = \{(1, 1, 1, 1, 1, 1, 1), (1, 2, 1, 1, 1, 1), (1, 3, 1, 1, 1), (1, 2, 2, 1, 1), (1, 4, 1, 1), (1, 3, 2, 1), (1, 2, 3, 1), (1, 5, 1), (1, 2, 2, 2), (1, 4, 2), (1, 3, 3), (1, 2, 4), (1, 6)\};
\]

\[
F(8) = \{(1, 1, 1, 1, 1, 1, 1, 1), (1, 2, 1, 1, 1, 1, 1), (1, 3, 1, 1, 1, 1), (1, 2, 2, 1, 1, 1), (1, 4, 1, 1, 1), (1, 3, 2, 1, 1), (1, 2, 3, 1, 1), (1, 5, 1, 1), (1, 2, 2, 2, 1), (1, 4, 2, 1), (1, 3, 3, 1), (1, 2, 4, 1), (1, 6, 1), (1, 3, 2, 2), (1, 2, 3, 2), (1, 5, 2), (1, 2, 2, 3), (1, 4, 3), (1, 3, 4), (1, 2, 5), (1, 7)\}.
\]

**Theorem 2.2.** For \( n \geq 1 \), the cardinality of \( F(n) \) is equal to the \( n \)th Fibonacci number \( F_n \) (see [2], Lemma 2.3).

**Theorem 2.3.** For all \( n \geq 1 \), \( M(n) \subseteq F(n) \). In particular, the sequence \( l(n) \) is bounded above by the Fibonacci sequence \( l(n) \leq F_n \) (see [2], Theorem 2.4).

3. Lucas Bound

**Definition 3.1.** For \( n \geq 1 \), define recursively the Lucas set \( L(n) \) of vectors with entries in \( \mathbb{N} \) as follows:

(1) \( L(1) = \{(1)\}; \ L(2) = \{(1, 1), (1, 2)\} \).

(2) For \( n \geq 3 \) define \( L(n) := C(n) \cup D(n) \) where

\[
C(n) := \{(1, t_1, \ldots, t_s, 1) \mid (1, t_1, \ldots, t_s) \in L(n - 1)\}
\]

\[
D(n) := \{(1, t_1, \ldots, t_{s-1}, t_s + 1) \mid (1, t_1, \ldots, t_s) \in L(n - 1), \text{ with } t_{s-1} > 1 \text{ or } s = 1\}.
\]

**Remark 3.2.** The sets \( C(n) \) and \( D(n) \) of Definition 3.1 form a disjoint set partition of \( L(n) \). The sets \( L(1), L(2), \ldots \), are mutually disjoint.

The first few sets \( L(n) \) are
For $n > 1$, the claim is true, since $|L(1)| = L_1 = 1$.

For $n = 2$, the claim is true, since $|L(2)| = L_2 = 3$.

Suppose the claim is true for all $r < k$, that is $|L(r)| = L_r$ for all $r < k$.

We have to prove that the claim is true for $n = k$, that is, $|L(k)| = L_k$.

We have $|L(k - 2)| = L_{k-2}$ and $|L(k - 1)| = L_{k-1}$.

From these equalities we have

$$|L(k - 2)| + |L(k - 1)| = L_{k-2} + L_{k-1} = L_k.$$
Hence
\[ |L(k)| = L_k. \]

**Theorem 3.4.** For all \( n \geq 2 \), \( M(n+1) \subseteq L(n) \). In particular, the sequence \( l(n+1) \) is bounded from above by the Lucas sequence.

**Proof.** By the construction one can check that \( L(n) \) is the set of all integer vectors \((1, t_1, \ldots, t_s)\) with \( 1 + t_1 + t_2 + \cdots + t_s = n + 1 \) and the property that if \( t_i = 1 \) then \( t_j = 1 \) for all \( j \geq i \).

Let \( m = (1, m_1, \ldots, m_s) \in M(n+1) \) be an \( M \)-sequence of length \( n \).

Using Macaulay’s Theorem it is not hard to see that if \( m_i = 1 \) for some \( i \geq 1 \), then \( m_j = 1 \) for all \( j \geq i \). We prove this by induction on \( n \). We have
\[ M(n+1) \subseteq L(n), n \geq 2, \]
For \( n = 2 \), the claim is true, since \( M(3) \subseteq L(2) \).
\[ M(3) = \{(1,1,1), (1,2)\} \quad \text{and} \quad L(2) = \{(1,1,1), (1,2)\}. \]
For \( n = 3 \), the claim is true, since \( M(4) \subseteq L(3) \).
\[ M(4) = \{(1,1,1,1), (1,2,1), (1,3)\} \quad \text{and} \quad L(3) = \{(1,1,1,1), (1,1), (1,2,1), (1,3)\}. \]
By induction we assume that \( M(k) \subseteq L(k-1) \) and \( M(k+1) \subseteq L(k) \).

We have to prove that the claim is true for \( n = k + 1 \), that is, \( M(k+2) \subseteq L(k+1) \).

Let \( s(S) \) denote the number of elements of a set \( S \).

We have \( M(k) \subseteq L(k-1) \Rightarrow s(M(k)) \leq s(L(k-1)) \),
\[ M(k+1) \subseteq L(k) \Rightarrow s(M(k+1)) \leq s(L(k)) , \]
\[ s(M(k) \cup M(k+1)) = s(M(k)) + s(M(k+1)) - s(M(k) \cap M(k+1)) , \]
\[ s(M(k) \cap M(k+1)) = 0 , \]
and hence
\[ s(M(k) \cup M(k+1)) = s(M(k)) + s(M(k+1)) . \]

From these equalities we have
\[ M(k) \cup M(k+1) \subseteq L(k-1) \cup L(k) . \]
In the same way, we have
\[ s(L(k-1) \cup L(k)) = s(L(k-1)) + s(L(k)) - s(L(k-1) \cap L(k)) , \]
\[ s(L(k-1) \cap L(k)) = 0 , \]
\[ s(L(k-1) \cup L(k)) = s(L(k-1)) + s(L(k)) \]
and \[ s(M(k)) + s(M(k+1)) \leq s(L(k-1)) + s(L(k)) . \]

Hence
\[ s(M(k)) + s(M(k+1)) = s(M(k+2)) , s(L(k-1)) + s(L(k)) = s(L(k+1)) . \]

We know
\[ s(M(k+2)) \leq s(L(k+1)) . \]

Hence
\[ M(k+2) \subseteq L(k+1) . \]

Thus \( M(n+1) \subseteq L(n) \).

**Theorem 3.5.** For all \( n \geq 2 \), we have the relation
\[ L(n) \setminus M(n + 1) = M(n - 1) . \]

**Proof.** We prove this by induction on \( n \).
For \( n = 2 \), the claim is true, since \( L(2) \setminus M(3) = M(1) \).

For \( n = 3 \), the claim is true, since \( L(3) \setminus M(4) = M(2) \).

Suppose the claim is true for some \( n = k \geq 3 \), that is \( L(k) \setminus M(k + 1) = M(k - 1) \).

We have to prove that the claim is true for \( n = k + 1 \), that is,

\[
L(k + 1) \setminus M(k + 2) = M(k).
\]

The identity \( L(k) \setminus M(k + 1) = M(k - 1) \) implies \( L(k) = M(k - 1) \cup M(k + 1) \).

From these equalities, we can easily see that

\[
L(k + 1) \setminus M(k + 2) = M(k).
\]

\[ \square \]

**Corollary 3.6.** For all \( n \geq 2 \), we have \( |L(n)| - |F(n + 1)| = |F(n - 1)| \).

**Proof.** We prove that by induction on \( n \).

For \( n = 2 \), the claim \( |L(2)| - |F(3)| = |F(1)| \) is true, since \( |L(2)| = 3, |F(3)| = 2 \) and \( |F(1)| = 1 \).

Suppose the claim is true for all \( n \leq k \) for some \( k \geq 2 \), that is

\[
|L(n)| - |F(n + 1)| = |F(n - 1)|.
\]

We have to prove that the claim is true for \( n = k + 1 \), that is

\[
|L(k + 1)| - |F(k + 2)| = |F(k)|.
\]

Thus by induction we have

\[
|L(k - 1)| = |F(k - 2)| + |F(k)|
\]

\[
|L(k)| = |F(k - 1)| + |F(k + 1)|
\]

From these equalities, we have

\[
|L(k - 1)| + |L(k)| = |F(k - 2)| + |F(k - 1)| + |F(k)| + |F(k + 1)|.
\]

From the last equality,

\[
|L(k + 1)| - |F(k + 2)| = |F(k)|
\]

since \( |L(k - 1)| + |L(k)| = |L(k + 1)|, |F(k - 2)| + |F(k - 1)| = |F(k)| \) and

\[
|F(k)| + |F(k + 1)| = |F(k + 2)|.
\]

\[ \square \]

**References**


Engin Özkan
Erzincan University, Erzincan, Turkey
E-mail address: eozkan@erzincan.edu.tr

Aykut Geçer
Erzincan University, Erzincan, Turkey
E-mail address: aykut2057@hotmail.com

İpek Altun
Erzincan University, Erzincan, Turkey
E-mail address: ipekaltun21@hotmail.com