A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY DERIVATIVE OPERATOR

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ABSTRACT. In the present paper, we introduce a new subclass of harmonic univalent functions in the unit disc $U$ by using Derivative operator. Also, we obtain coefficient conditions, distortion bounds, convolution conditions, convex combinations, extreme points and discuss a class preserving integral operator. Relevant connections of the results presented here with various known results are briefly indicated.

1. INTRODUCTION

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a complex domain $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply-connected domain $D \subset C$, we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$, see [4].

Denote by $S_{H}(j)$, the class of functions $f = h + \overline{g}$ that are harmonic, univalent and sense-preserving in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ with normalization $f(0) = h(0) = f_{z}(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_{H}(j)$, we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{k=j+1}^{\infty} a_{k}z^{k}, \quad g(z) = \sum_{k=1}^{\infty} b_{k}z^{k}, \quad |b_{1}| < 1. \quad (1.1)$$

For $j = 1$ the class $S_{H}(j)$ reduce to the class $S_{H}$ of harmonic univalent functions in $U$ and for $j = 1, g \equiv 0$ it reduce to the class $S$ of normalized analytic univalent functions.

Al-Shaqsi and Darus [3] introduced the derivative operator for functions $f$ of the form (1.1) as:

$$D_{\lambda}^{n}f(z) = D_{\lambda}^{n}h(z) + (-1)^{n}D_{\lambda}^{n}\overline{g(z)}, \; n, \lambda \in N_{0} = N \cup \{0\}, \; z \in U, \quad (1.2)$$

where

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obtain the following known subclasses studied earlier by various researchers.

By assigning specific values to $n, \lambda, j, \alpha, \rho$ we have

$$D_\lambda^h(z) = z + \sum_{k=j+1}^{\infty} k^n C(\lambda, k) a_k z^k, \quad D_\lambda^g(z) = \sum_{k=1}^{\infty} k^n C(\lambda, k) b_k z^k$$

and $C(\lambda, k) = (k+\lambda^{-1})$.

It is easy to see that for $\lambda = 0$ the operator $D_\lambda^h$ reduce to the modified Salagean derivative operator introduced by Jahangiri et al. [6].

Now we introduce the class $G_H(n, \lambda, j, \alpha, \rho, t)$ of functions of the form (1.1) that satisfy the following condition

$$\Re \left\{ (1 + pe^{i\eta}) \frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f(z)} - pe^{i\eta} \right\} > \alpha,$$

where $0 \leq \alpha < 1, \eta \in R, \rho \geq 0, j \in N, n, \lambda \in N_0, 0 \leq t \leq 1, f(z) = (1-t)z + tf(z)$ and $D_\lambda^n f(z)$ is defined by (1.2).

Let $\mathcal{G}_H(n, \lambda, j, \alpha, \rho, t)$ denote the subclass of $G_H(n, \lambda, j, \alpha, \rho, t)$ consisting of harmonic functions $f_n = h + \mathcal{G}_n$ such that $h$ and $\mathcal{G}_n$ are of the form

$$h(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k, \quad \mathcal{G}_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k.$$

Assigning specific values to $n, \lambda, j, \alpha, \rho$ and $t$ in the subclass $G_H(n, \lambda, j, \alpha, \rho, t)$, we obtain the following known subclasses studied earlier by various researchers.

(i) $G_H(n, \lambda, 1, \alpha, \rho, t)$ studied by Pathak et al. [9].
(ii) $G_H(n, \lambda, 1, \alpha, 0, t)$ studied by Al-Shaqi and Darus [3].
(iii) $G_H(n, 0, 1, \alpha, 1, t)$ studied by Yalcin et al. [13].
(iv) $G_H(0, \lambda, 1, \alpha, 0, t)$ studied by Murugusundaramoorthy and Vijaya [8].
(v) $G_H(0, 1, 1, \alpha, 1, t)$ studied by Rosy et al. [10].
(vi) $G_H(0, 0, 1, \alpha, 0, t)$ studied by Jahangiri [5].
(vii) $G_H(1, 0, 1, \alpha, 0, t)$ studied by Jahangiri [5].
(viii) $G_H(0, 0, 1, \alpha, 0, 1)$ studied by Jahangiri et al. [6].
(ix) $G_H(1, 0, 1, \alpha, 1, 1)$ studied by Kim et al. [7].
(x) $G_H(0, 0, 1, \alpha, 1, 1)$ studied by Ahuja et al. [2].
(xi) $G_H(1, 0, 1, \alpha, 0, 0)$ studied by Ahuja and Jahangiri [1].

In the present paper, we obtain coefficient condition, distortion bound, extreme points, convolution and convex combination. Finally we discuss a class preserving integral operator for this class.

2. Coefficient Bound

We begin with a sufficient coefficient condition for functions in $G_H(n, \lambda, j, \alpha, \rho, t)$.

Theorem 2.1. Let $f = h + \mathcal{G}$ be given by (1.1). If

$$\sum_{k=j+1}^{\infty} \{k(1+\rho) - t(\alpha + \rho)\} |a_k| k^n C(\lambda, k) + \sum_{k=1}^{\infty} \{k(1+\rho) + t(\alpha + \rho)\} |b_k|$$

$$k^n C(\lambda, k) \leq 1 - \alpha,$$ (2.1)
where $n, \lambda \in \mathbb{N}_0, j \in \mathbb{N}, C(\lambda, k) = (k^\lambda + 1)$, $\rho \geq 0$, $0 \leq t \leq 1$ and $0 \leq \alpha < 1$, then $f$ is sense-preserving, harmonic univalent in $U$ and $f \in G_H(n, \lambda, j, \rho, t)$.

Proof. If $z_1 \neq z_2$, then

$$
\frac{|f(z_1) - f(z_2)|}{|h(z_1) - h(z_2)|} \geq 1 - \frac{|g(z_1) - g(z_2)|}{|h(z_1) - h(z_2)|} = 1 - \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=j+1}^{\infty} a_k (z_1^k - z_2^k)}
$$

$$
> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{\sum_{k=j+1}^{\infty} k|a_k|}
$$

$$
> 1 - \frac{\sum_{k=1}^{\infty} [k(1 + \rho) + t(\alpha + \rho)]k^{n}C(\lambda, k)|b_k|}{\sum_{k=j+1}^{\infty} [k(1 + \rho) - t(\alpha + \rho)]k^{n}C(\lambda, k)|a_k|}
$$

$$
\geq 1 - \frac{\sum_{k=1}^{\infty} [k(1 + \rho) + t(\alpha + \rho)]k^{n}C(\lambda, k)|b_k|}{1 - \alpha}
$$

$$
\geq 0
$$

which proves univalence. Note that $f$ is sense-preserving in $U$. This is because

$$
|h'(z)| \geq 1 - \sum_{k=j+1}^{\infty} k|a_k||z|^{k-1}
$$

$$
> 1 - \sum_{k=j+1}^{\infty} \frac{[k(1 + \rho) - t(\alpha + \rho)]k^{n}C(\lambda, k)|a_k|}{1 - \alpha}
$$

$$
\geq \sum_{k=1}^{\infty} \frac{[k(1 + \rho) + t(\alpha + \rho)]k^{n}C(\lambda, k)|b_k|}{1 - \alpha}
$$

$$
\geq \sum_{k=1}^{\infty} \frac{[k(1 + \rho) + t(\alpha + \rho)]k^{n}C(\lambda, k)|b_k||z|^{k-1}}{1 - \alpha}
$$

$$
\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.
$$
Using the fact that $\Re w > \alpha$, if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$ it suffices to show that

$$
|(1 - \alpha) + (1 + \rho e^{i\eta}) \frac{D_{\lambda}^{n+1} f(z)}{D_{\lambda}^{n} f(z)} - \rho e^{i\eta}| - |(1 + \alpha) - (1 + \rho e^{i\eta}) \frac{D_{\lambda}^{n+1} f(z)}{D_{\lambda}^{n} f(z)} + \rho e^{i\eta}| \geq 0. \tag{2.4}
$$

Substituting the values of $D_{\lambda}^{n+1} f(z)$ and $D_{\lambda}^{n} f(z)$ in (2.4), we obtain

$$
|(1 - \alpha - \rho e^{i\eta})D_{\lambda}^{n} f(z) + (1 + \rho e^{i\eta})D_{\lambda}^{n+1} f(z)| - \\
| - (1 + \alpha + \rho e^{i\eta})D_{\lambda}^{n} f(z) + (1 + \rho e^{i\eta})D_{\lambda}^{n+1} f(z)|
$$

$$
= |(2 - \alpha)z + \sum_{k=j+1}^{\infty} \{k(1 + \rho e^{i\eta}) + t(1 - \alpha - \rho e^{i\eta})\}k^n C(\lambda, k) a_k z^k - a_k z^k + (-1)^n \sum_{k=j+1}^{\infty} \{k(1 + \rho e^{i\eta}) - t(1 - \alpha + \rho e^{i\eta})\}k^n C(\lambda, k)b_k z^k - \\
- | - \alpha z + \sum_{k=j+1}^{\infty} \{k(1 + \rho e^{i\eta}) - t(1 + \alpha + \rho e^{i\eta})\}k^n C(\lambda, k)a_k z^k - a_k z^k - (-1)^n \sum_{k=1}^{\infty} \{k(1 + \rho e^{i\eta}) + t(1 + \alpha + \rho e^{i\eta})\}k^n C(\lambda, k)b_k z^k |
$$

$$
\geq 2(1 - \alpha)|z| \left[ 1 - \sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)\}k^n C(\lambda, k)|a_k||z|^{k-1}}{1 - \alpha} - \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)\}k^n C(\lambda, k)|b_k||z|^{k-1}}{1 - \alpha} \right]
$$

$$
> 2(1 - \alpha)|z| \left[ 1 - \sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)\}k^n C(\lambda, k)|a_k|}{1 - \alpha} - \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)\}k^n C(\lambda, k)|b_k|}{1 - \alpha} \right]. \tag{2.5}
$$

This last expressions is non-negative by (2.1), and so the proof is completed.

The harmonic function

$$
f(z) = z + \sum_{k=j+1}^{\infty} \frac{(1 - \alpha)}{k(1 + \rho) - t(\alpha + \rho)\}k^n C(\lambda, k)|a_k|}x_k z^k + \sum_{k=1}^{\infty} \frac{(1 - \alpha)}{k(1 + \rho) + t(\alpha + \rho)\}k^n C(\lambda, k)|b_k|}y_k z^k \tag{2.6}
$$
where \( n, \lambda \in \mathbb{N}_0, j \in \mathbb{N}, 0 \leq t \leq 1, \rho \geq 0, 0 \leq \alpha < 1 \) and \( \sum_{k=j+1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1 \) shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.6) are in \( G_H(n, \lambda, j, \alpha, \rho, t) \) because

\[
\sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |a_k| + \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |b_k| = \sum_{k=j+1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1. \tag{2.7}
\]

In the following theorem, it is shown that the condition (2.1) is also necessary for functions \( f_n = h + g_n \), where \( h \) and \( g_n \) are of the form (1.4).

**Theorem 2.2.** Let \( f_n = h + g_n \) be given by (1.4). Then \( f_n \in G_H(n, \lambda, j, \alpha, \rho, t) \), if and only if

\[
\sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |a_k| + \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |b_k| \leq 1, \tag{2.8}
\]

where \( n, \lambda \in \mathbb{N}_0, j \in \mathbb{N}, C(\lambda, k) = \binom{k+\lambda-1}{\lambda}, \rho \geq 0, 0 \leq \alpha < 1, 0 \leq t \leq 1 \).

**Proof.** Since \( G_H(n, \lambda, j, \alpha, \rho, t) \subset G_H(n, \lambda, j, \alpha, \rho, t) \), we only need to prove the “only if” part of the theorem. To this end, for functions \( f_n \) of the form (1.4), we
notice that the condition \((1.3)\) is equivalent to

\[
\Re \left\{ \frac{(1 + \rho e^{i\eta}) D_{\lambda}^{n+1} f(z)}{D_{\lambda}^{n} f(z)} - (\rho e^{i\eta} + \alpha) \right\} \geq 0
\]

\[
\Rightarrow \Re \left\{ \frac{(1 + \rho e^{i\eta}) D_{\lambda}^{n+1} f(z) - (\rho e^{i\eta} + \alpha) D_{\lambda}^{n} f(z)}{D_{\lambda}^{n} f(z)} \right\} \geq 0
\]

\[
\Rightarrow \Re \left\{ \left(1 + \rho e^{i\eta}\right) \left(z - \sum_{k=j+1}^{\infty} k^{n+1} C(\lambda, k) |a_k| z^k + (-1)^{2n+1} \sum_{k=1}^{\infty} k^{n+1} |b_k| C(\lambda, k) \frac{z^k}{k} \right) - (\rho e^{i\eta} + \alpha) \left(z - \sum_{k=j+1}^{\infty} k^n C(\lambda, k) t|a_k| z^k + (-1)^{2n} \sum_{k=1}^{\infty} k^n t|b_k| C(\lambda, k) \frac{z^k}{k} \right) \right\} \geq 0
\]

\[
\Rightarrow \Re \left\{ \left(1 - \alpha\right) z - \sum_{k=j+1}^{\infty} k^n \left(1 + \rho e^{i\eta}\right) - t \left(\rho e^{i\eta} + \alpha\right) C(\lambda, k) |a_k| z^k + (-1)^{2n+1} \sum_{k=1}^{\infty} k^n |b_k| C(\lambda, k) \frac{z^k}{k} \right\} \geq 0
\]

\[
\Rightarrow \Re \left\{ \left(1 - \alpha\right) - \sum_{k=j+1}^{\infty} k^n \left(1 + \rho e^{i\eta}\right) - t \left(\rho e^{i\eta} + \alpha\right) C(\lambda, k) |a_k| z^{k-1} - \sum_{k=j+1}^{\infty} k^n C(\lambda, k) t|a_k| z^{k-1} + \frac{z}{n} (-1)^{2n} \sum_{k=1}^{\infty} k^n C(\lambda, k) t|b_k| z^{k-1} - \sum_{k=j+1}^{\infty} k^n C(\lambda, k) t|b_k| z^{k-1} \right\} \geq 0
\]
The above condition \[ (2.9) \] must hold for all values of \( z \) on the positive real axes, where, \( 0 \leq |z| = r < 1 \), we must have

\[
\Re \left\{ \left( 1 - \sum_{k=j+1}^{\infty} k^n (k-t\alpha)C(\lambda, k) |a_k|^r k^{-1} - (-1)^2n \sum_{k=1}^{\infty} k^n (k+t\alpha)C(\lambda, k) |b_k|^r k^{-1} \right) \right\} \geq 0
\]

Since \( \Re(-e^{i\eta}) \geq -|e^{i\eta}| = -1 \), the above inequality reduce to

\[
\left\{ (1 - \sum_{k=j+1}^{\infty} k^n \{k(1+\rho) - t(\rho + \alpha)\} C(\lambda, k) |a_k|^r k^{-1} - \sum_{k=1}^{\infty} k^n \{k(1+\rho) + t(\rho + \alpha)\} C(\lambda, k) |b_k|^r k^{-1} \right\}^{-1} \geq 0. \tag{2.10}
\]

If the condition \[ (2.8) \] does not hold, then the numerator in \[ (2.10) \] is negative for \( r \) sufficiently close to 1. Hence there exist \( a, z_0 = r_0 \) in \((0,1)\) for which the quotient in \[ (2.10) \] is negative. This contradicts the condition for \( f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \) and so the proof is complete.

\[ \square \]

3. Distortion Bounds

In this section, we will obtain distortion bounds for functions in \( \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \).

**Theorem 3.1.** Let \( f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \). Then for \( |z| = r < 1 \), we have

\[
|f_n(z)| \leq (1 + |b_1| + |b_2| |r| + \ldots + |b_j||r^{j-1}|)^r + \frac{1 - \alpha}{(j+1)(1+\rho) - t(\rho + \alpha)} C(\lambda,j+1) \frac{1}{1 - \alpha} |b_j| j^n C(\lambda,j) \] \[ (j+1) \]

\[
|f_n(z)| \geq (1 - |b_1| - |b_2| |r| - \ldots - |b_j||r^{j-1}|)^r - \frac{1 - \alpha}{(j+1)(1+\rho) - t(\rho + \alpha)} C(\lambda,j+1) \frac{1}{1 - \alpha} |b_j| j^n C(\lambda,j) \] \[ (j+1) \]
Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and is thus omitted. Let \( f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \). Taking the absolute value of \( f_n \), we obtain

\[
|f_n(z)| = \left| z - \sum_{k=j+1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k z^k \right| \leq (1 + |b_1| + |b_2| r + \ldots + |b_j| r^{j-1}) r + \sum_{k=j+1}^{\infty} (|a_k| + |b_k|) r^k
\]

\[
\leq (1 + |b_1| + |b_2| r + \ldots + |b_j| r^{j-1}) r + \sum_{k=j+1}^{\infty} (|a_k| + |b_k|) r^{j+1}
\]

\[
\leq (1 + |b_1| + |b_2| r + \ldots + |b_j| r^{j-1}) r + \sum_{k=j+1}^{\infty} \frac{1 - \alpha}{((j + 1) (1 + \rho) - t(\rho + \alpha)) (j + 1)^n C(\lambda, j + 1)} |a_k|
\]

\[
+ \frac{1 - \alpha}{((j + 1) (1 + \rho) - t(\rho + \alpha)) (j + 1)^n C(\lambda, j + 1)} |b_k| r^{(j+1)}
\]

\[
\leq (1 + |b_1| + |b_2| r + \ldots + |b_j| r^{j-1}) r + \sum_{k=j+1}^{\infty} \frac{1 - \alpha}{((j + 1) (1 + \rho) - t(\rho + \alpha)) (j + 1)^n C(\lambda, j + 1)} |a_k| + \frac{1 - \alpha}{((j + 1) (1 + \rho) - t(\rho + \alpha)) (j + 1)^n C(\lambda, j + 1)} |b_k| r^{(j+1)}
\]

\[
\leq (1 + |b_1| + |b_2| r + \ldots + |b_j| r^{j-1}) r + \left[ 1 - \frac{(1 + t + 1) \rho + \alpha}{1 - \alpha} |b_1| - \frac{(2(1 + \rho) + t(\rho + \alpha)}{1 - \alpha} |b_2| 2^n C(\lambda, 2) \ldots - \frac{(j(1 + \rho) + t(\rho + \alpha))}{1 - \alpha} |b_j| j^n C(\lambda, j) \right] r^{(j+1)}
\]

\[
\square
\]

4. Convolution, Convex Combination and Extreme Points

In this section, we show that the class \( \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \) is invariant under convolution and convex combination.

For harmonic functions

\[
f_n(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k
\]

and

\[
F_n(z) = z - \sum_{k=j+1}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| z^k,
\]
the convolution of $f_n$ and $F_n$ is given by

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=j+1}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| z^k. \quad (4.1)$$

**Theorem 4.1.** For $0 \leq \beta \leq \alpha < 1$, $n, \lambda \in \mathbb{N}_0, j \in \mathbb{N}, \rho \geq 0$, $0 \leq t \leq 1$ let $f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$ and $F_n \in \overline{G}_H(n, \lambda, j, \beta, \rho, t)$. Then $f_n * F_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \subset \overline{G}_H(n, \lambda, j, \beta, \rho, t)$.

**Proof.** We wish to show that the coefficient of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{G}_H(n, \lambda, j, \beta, \rho, t)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |a_k||A_k|$$

$$+ \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |b_k||B_k|$$

$$\leq \sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |b_k|$$

$$\leq \sum_{k=j+1}^{\infty} \frac{k(1 + \rho) - t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{k(1 + \rho) + t(\alpha + \rho)}{1 - \alpha} k^n C(\lambda, k) |b_k|$$

$$\leq 1.$$  

Since $0 \leq \beta \leq \alpha < 1$ and $f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$. Therefore $f_n * F_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \subset \overline{G}_H(n, \lambda, j, \beta, \rho, t)$.

We now examine the convex combination of $\overline{G}_H(n, \lambda, j, \alpha, \rho, t)$.

Let the functions $f_{n_i}(z)$ be defined, for $i = 1, 2, \ldots, m$, by

$$f_{n_i}(z) = z - \sum_{k=j+1}^{\infty} |a_{k,i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,i}| z^k. \quad (4.2)$$

\[\square\]
Theorem 4.2. Let the functions \( f_n(z) \) defined by (4.2) be in the class \( G_H(n, \lambda, j, \alpha, \rho, t) \) for every \( i = 1, 2, \ldots, m \). Then the functions \( t_i(z) \) defined by
\[
t_i(z) = \sum_{i=1}^{m} c_i f_n(z), \quad 0 \leq c_i \leq 1
\]
are also in the class \( G_H(n, \lambda, j, \alpha, \rho, t) \), where \( \sum_{i=1}^{m} c_i = 1 \).

Proof. According to the definition of \( t_i \), we can write
\[
t_i(z) = z - \sum_{k=j+1}^{\infty} \left( \sum_{i=1}^{m} c_i |a_{k,i}| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{m} c_i |b_{k,i}| \right) z^k.
\]
Further, since \( f_n(z) \) are in \( G_H(n, \lambda, j, \alpha, \rho, t) \) for every \( i = 1, 2, \ldots, m \), then
\[
\sum_{k=j+1}^{\infty} \left[ (k(1+\rho) - t(\alpha + \rho)) \left( \sum_{i=1}^{m} c_i |a_{k,i}| \right) + \sum_{k=1}^{\infty} (k(1+\rho) + t(\alpha + \rho)) \left( \sum_{i=1}^{m} c_i |b_{k,i}| \right) \right] k^n C(\lambda, k) \leq \sum_{i=1}^{m} c_i (1 - \alpha) \leq (1 - \alpha).
\]
Hence the Theorem 4.2 follows.

Next we determine the extreme points of closed convex hulls of \( G_H(n, \lambda, j, \alpha, \rho, t) \) denoted by \( \text{clco} \ G_H(n, \lambda, j, \alpha, \rho, t) \).

Theorem 4.3. Let \( f_n \) be given by (1.4). Then \( f_n \in G_H(n, \lambda, j, \alpha, \rho, t) \), if and only if
\[
f_n(z) = \sum_{k=j}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z),
\]
where
\[
h_j(z) = z, \quad h_k(z) = z - \left( \frac{1 - \alpha}{(k(1+\rho) - t(\alpha + \rho)) k^n C(\lambda, k)} \right) z^k, \quad k = j + 1, j + 2, \ldots.
\]
\[
g_{n_k}(z) = z + (-1)^n \left( \frac{1 - \alpha}{(k(1+\rho) + t(\alpha + \rho)) k^n C(\lambda, k)} \right) z^k, \quad k = 1, 2, 3, \ldots.
\]
and \( \sum_{k=j}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1, \quad X_k \geq 0, \quad Y_k \geq 0 \). In particular, the extreme points of \( G_H(n, \lambda, j, \alpha, \rho, t) \) are \( \{h_k\} \) and \( \{g_{n_k}\} \).
Proof. For the function $f_n$ of the form (4.4) we have

$$f_n(z) = \sum_{k=j}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{nk}(z)$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1 - \alpha}{(k(1 + \rho) - t(\alpha + \rho))k^nC(\lambda, k)} X_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{1 - \alpha}{(k(1 + \rho) + t(\alpha + \rho))k^nC(\lambda, k)} Y_k z^k$$

Then

$$\sum_{k=j+1}^{\infty} \left( \frac{k(1 + \rho) - t(\alpha + \rho))k^nC(\lambda, k)}{1 - \alpha} \right) |a_k| + \sum_{k=1}^{\infty} \left( \frac{k(1 + \rho) + t(\alpha + \rho))k^nC(\lambda, k)}{1 - \alpha} \right) |b_k|$$

$$= \sum_{k=j+1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$$

$$= 1 - X_j \leq 1$$

and so $f_n \in \text{clco} \ G_H(n, \lambda, j, \alpha, \rho, t)$.

Conversely, suppose that $f_n \in \text{clco} \ G_H(n, \lambda, j, \alpha, \rho, t)$. Setting

$$X_k = \left( \frac{1 - \alpha}{(k(1 + \rho) - t(\alpha + \rho))k^nC(\lambda, k)} \right) |a_k|, \quad 0 \leq X_k \leq 1 \quad k = j + 1, j + 2, \ldots$$

$$Y_k = \left( \frac{1 - \alpha}{(k(1 + \rho) + t(\alpha + \rho))k^nC(\lambda, k)} \right) |b_k|, \quad 0 \leq Y_k \leq 1 \quad k = 1, 2, 3, \ldots$$

(4.5)

and $X_j = 1 - \sum_{k=j+1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$. Therefore, $f_n$ can be written as

$$f_n(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| z^k$$

$$= z - \sum_{k=j+1}^{\infty} \left( \frac{1 - \alpha}{(k(1 + \rho) - t(\alpha + \rho))k^nC(\lambda, k)} \right) X_k z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} \left( \frac{1 - \alpha}{(k(1 + \rho) + t(\alpha + \rho))k^nC(\lambda, k)} \right) Y_k z^k$$

$$= z + \sum_{k=j+1}^{\infty} (h_k(z) - z) X_k + \sum_{k=1}^{\infty} (g_{nk}(z) - z) Y_k$$

$$= \sum_{k=j+1}^{\infty} h_k(z) X_k + \sum_{k=1}^{\infty} g_{nk}(z) Y_k + z \left( 1 - \sum_{k=j+1}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \right)$$

$$= \sum_{k=j}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{nk}(z), \quad \text{as required.}$$

(4.6)

This completes the proof of Theorem 4.3.
5. A FAMILY OF CLASS PRESERVING INTEGRAL OPERATOR

Let \( f(z) = h(z) + g(z) \) be defined by (1.1) then \( F(z) \) defined by the relation

\[
F(z) = c + \frac{1}{z} \int_0^z t^{c-1} h(t) dt + \frac{c+1}{z} \int_0^z t^{c-1} g(t) dt, \quad (c > -1).
\] (5.1)

**Theorem 5.1.** Let \( f_n(z) = h(z) + g_n(z) \in S_H \) be given by (1.4) and \( f_n(z) \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \) then \( F(z) \) be defined by (5.1) also belong to \( \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \).

**Proof.** From the representation of (5.1) of \( F(z) \), it follows that

\[
F(z) = z - \sum_{k=1}^{\infty} \left( \frac{c+1}{c+k} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} b_k z^k \right).
\] (5.2)

Since \( f_n(z) \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \), then by Theorem 2.2 we have

\[
\sum_{k=1}^{\infty} \frac{(k(1+\rho) - t(\alpha + \rho))k^n C(\lambda, k)}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho) + t(\alpha + \rho))k^n C(\lambda, k)}{1 - \alpha} |b_k| \leq 1.
\]

Now

\[
\sum_{k=2}^{\infty} \frac{(k(1+\rho) - t(\alpha + \rho))k^n C(\lambda, k)}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho) + t(\alpha + \rho))k^n C(\lambda, k)}{1 - \alpha} |b_k|
\]

\[
\leq \sum_{k=2}^{\infty} \frac{(k(1+\rho) - t(\alpha + \rho))k^n C(\lambda, k)}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho) + t(\alpha + \rho))k^n C(\lambda, k)}{1 - \alpha} |b_k|
\]

\[
\leq 1.
\]

Thus \( F(z) \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \). \qed

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