CERTAIN SUBCLASS OF \( p\)-VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH LIU-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce the class \( \Lambda_{p,q,s}(\alpha_1;\gamma,\varrho,\beta) \) of meromorphic \( p\)\( -\)valent functions of order \( \gamma \)\( (0 \leq \gamma < p) \) and types \( \beta \) in the punctured unit disc \( U^* \) which are defined by making use of Liu-Srivastava operator. We investigate various properties and characteristics of this class.

1. Introduction

The class of meromorphic functions which are analytic and \( p\)\( -\)valent in \( U^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = \mathbb{U}\backslash\{0\} \) and has the form:

\[
f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p, n \in \mathbb{N}, p < n),
\]

is denoted by \( \sum_p \). For the function \( f(z) \) in this form and \( g(z) \in \sum_p \) given by

\[
g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \quad (p \in \mathbb{N}),
\]

the Hadamard products (or convolution) of \( f(z) \) and \( g(z) \) is given by

\[
(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_{n-p} z^{n-p} = (g * f)(z).
\]

For complex numbers \( \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_s \) \( (\beta_j \notin \mathbb{Z}^- = \{0,-1,-2,\ldots\}; j = 1,2,\ldots,s) \), Liu and Srivastava \cite{8} considered the linear operator

\[
M_{p,q,s}(\alpha_1)f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n a_{n-p}}{(\beta_1)_n \cdots (\beta_q)_n n!} z^{n-p},
\]

where \( \theta+\mu \) is the Pochhammer symbol defined by

\[
(\theta)_{\mu} = \frac{\Gamma(\theta+\mu)}{\Gamma(\theta)} = \begin{cases} 1 & (\mu = 0; \theta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ \theta(\theta+1) \cdots (\theta+\mu-1) & (\mu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}
\]
Proof. A function \( f \in \sum_p \) is in the class \( \Lambda_{p,q,s}(\alpha_1; \gamma, \varrho, \beta) \) if it also satisfies:

\[
\left| \frac{z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + p}{(2\varrho - 1)z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + (2\varrho \gamma - p)} \right| < \beta \quad (z \in U^*). \tag{6}
\]

Let \( \sum_p^* \) be the class of missing functions of the form:

\[
f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n| z^n \quad (p \in \mathbb{N}; z \in U^*) . \tag{7}
\]

Furthermore, we say that a function \( f(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, \varrho, \beta) \) when \( f(z) \) is of the form \( \text{(7)} \) and satisfies \( \text{(6)} \).

We note that:

\( \Lambda_{p,q,s}(\alpha_1; 1, 1, 1) = \Lambda_{p,q,s}^+(\alpha_1; 1, -1, \lambda) \) (see Aouf [3]).

For more details of meromorphic multivalent functions see [1,2,4,5,6,7,9,10].

In this paper, we investigate various important properties and characteristics of the class \( \Lambda_{p,q,s}^+(\alpha_1; \gamma, \varrho, \beta) \).

2. Properties of the Class \( \Lambda_{p,q,s}^+(\alpha_1; \gamma, \varrho, \beta) \)

In the reminder of this paper, we assume that: \( \alpha_j > 0 (j = 1, \ldots, q), \beta_j > 0 (j = 1, \ldots, s), 0 \leq \gamma < p, 0 < \beta \leq \frac{1}{2}, \varrho \leq 1, p \in \mathbb{N}, z \in U^* \).

**Theorem 1.** The function \( f(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, \varrho, \beta) \) if and only if

\[
\sum_{n=p}^{\infty} n[1 + \beta(2\varrho - 1)]\Gamma_{n+p}(\alpha_1) |a_n| \leq 2\beta \varrho(p - \gamma), \tag{8}
\]

where for convenience

\[
\Gamma_c(\alpha_1) = \frac{(\alpha_1)c(\alpha_2)c \ldots (\alpha_q)c}{(\beta_1)c(\beta_2)c \ldots (\beta_s)c}, \tag{9}
\]

**Proof.** Let \( f(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, \varrho, \beta) \). Then from \( \text{(6)} \) and \( \text{(7)} \), we have

\[
\left| \frac{z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + p}{(2\varrho - 1)z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + (2\varrho \gamma - p)} \right| = \left| \sum_{n=p}^{\infty} n\Gamma_{n+p}(\alpha_1) |a_n| z^{n+p} \right| < \beta. \tag{10}
\]

Since \( Re(z) \leq |z| (z \in \mathbb{C}) \), choosing \( z \) to be real and letting \( z \to 1^- \), then \( \text{(10)} \) yields

\[
\sum_{n=p}^{\infty} n\Gamma_{n+p}(\alpha_1) |a_n| \leq 2\beta \varrho(p - \gamma) - \sum_{n=p}^{\infty} n\beta(2\varrho - 1)\Gamma_{n+p}(\alpha_1) |a_n| , \tag{11}
\]

which leads to \( \text{(8)} \).
In order to prove the converse, we assume that the inequality (8) holds. Then, if we let \( z \in \partial U \), we find from (7) and (8) that

\[
|z^{p+1}(M_{p,q,s}(\alpha_1)f(z))' + p| \leq 2\beta(z \notin \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}).
\]  

Hence, by the maximum modulus theorem, we have

\[
|a_n| \leq \frac{2\beta \phi(p - \gamma)}{n[1 + \beta(2\phi - 1)]\Gamma_{n+p}(\alpha_1)} (n \geq p).
\]

The result is sharp for \( f \) given by

\[
f(z) = z^{-p} + \frac{2\beta \phi(p - \gamma)}{n[1 + \beta(2\phi - 1)]\Gamma_{n+p}(\alpha_1)} z^n (n \geq p).
\]

Next we prove the growth and distortion properties for the class \( \Lambda_{p,q,s}^+(\alpha_1; \gamma, q, \beta) \).

**Theorem 3.** If a function \( f(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, q, \beta) \) and the sequence \( \{D_n\} \) is nondecreasing, then

\[
|f^{(m)}(z)| \leq \left( \frac{(p + m - 1)!}{(p - 1)!} - \frac{p!}{(p - m)!} \right) \frac{(p - \gamma)}{D_p} r^{-(p-m)}
\]

\[
< \beta(z \notin \partial U = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}).
\]  

Hence, by the maximum modulus theorem, we have \( f(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, q, \beta) \). This completes the proof of Theorem [1].

**Corollary 2.** If \( f(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, q, \beta) \), then

\[
|a_n| \leq \frac{2\beta \phi(p - \gamma)}{n[1 + \beta(2\phi - 1)]\Gamma_{n+p}(\alpha_1)} (n \geq p).
\]

Equality holds for

\[
f(z) = z^{-p} + \frac{2\beta \phi(p - \gamma)}{n[1 + \beta(2\phi - 1)]\Gamma_{2p}(\alpha_1)} z^p.
\]

**Proof.** In view of Theorem [1], we have

\[
\frac{D_p}{p!} \sum_{n=p}^{\infty} n! |a_n| \leq \sum_{n=p}^{\infty} D_n |a_n| \leq p - \gamma,
\]

which yields

\[
\sum_{n=p}^{\infty} n! |a_n| \leq \frac{2p! \phi(p - \gamma)}{D_p}.
\]

Now, differentiating both sides of (7) \( m \)-times with respect to \( z \), we have

\[
f^{(m)}(z) = (-1)^m \frac{(p + m - 1)!}{(p - 1)!} z^{-(p+m)} + \sum_{n=p}^{\infty} \frac{n!}{(n - m)!} |a_n| z^{n-m},
\]
From (18) and (19), we have

\[ |f^{(m)}(z)| \geq \left( \frac{(p + m - 1)!}{(p - m)!} - \frac{p!}{D_p} \right) r^{-(p + m)} \]

and

\[ |f^{(m)}(z)| \geq \left( \frac{(p + m - 1)!}{(p - m)!} - \frac{p!}{D_p} \right) r^{-(p + m)} \]

This completes the proof of Theorem 3.

\[ \square \]

**Theorem 4.** Let \( f(z) \in \Lambda_{p,\gamma,\beta}(\alpha_1; \gamma, \beta, \gamma). \) Then

(i) \( f(z) \) is meromorphically \( p \)-valent starlike of order \( \delta(0 \leq \delta < p) \) in the disk \( |z| < r_1 \), where

\[ r_1 = \inf_{n \geq p} \left\{ \frac{n(p - \delta) [1 + \beta (2q - 1)] \Gamma_{n + p}(\alpha_1)}{2 \beta \varrho (p - \gamma) [(p + n) + (p - \delta)]} \right\}^{\frac{1}{n+p}}, \quad (20) \]

(ii) \( f(z) \) is meromorphically \( p \)-valent convex of order \( \delta(0 \leq \delta < p) \) in the disk \( |z| < r_2 \), where

\[ r_2 = \inf_{n \geq p} \left\{ \frac{(p - \delta) [1 + \beta (2q - 1)] \Gamma_{n + p}(\alpha_1)}{2 \beta \varrho (p - \gamma) [(p + n) + (p - \delta)]} \right\}^{\frac{1}{n+p}}. \quad (21) \]

Each of these results is sharp for the function \( f(z) \) given by (14).

**Proof.** (i) From (7), we easily get

\[ \left| \frac{z f'(z)}{f(z)} + p \right| \leq \frac{\infty \sum_{n=p}^{\infty} (p + n) \, |a_n| \, z^{n+p}}{1 - \sum_{n=p}^{\infty} |a_n| \, z^{n+p}}. \quad (22) \]

Thus,

\[ \left| \frac{z f'(z)}{f(z)} + p \right| \leq p - \delta (0 \leq \delta < p) \quad (23) \]

if

\[ \sum_{n=p}^{\infty} \frac{(p + n) + (p - \delta)}{(p - \delta)} |a_n| |z|^{n+p} \leq 1. \quad (24) \]

Hence, by Theorem 1 \( (14) \) will be true if

\[ \frac{(p + n) + (p - \delta)}{(p - \delta)} |z|^{n+p} \leq \frac{n [1 + \beta (2q - 1)] \Gamma_{n + p}(\alpha_1)}{2 \beta \varrho (p - \gamma)}, \quad (25) \]

that is if

\[ |z|^{n+p} \leq \frac{n(p - \delta) [1 + \beta (2q - 1)] \Gamma_{n + p}(\alpha_1)}{2 \beta \varrho (p - \gamma) [(p + n) + (p - \delta)]}. \]
(ii) Also from (7), we have

\[
1 + z\frac{f''(z)}{f'(z)} + p \leq \frac{\sum_{n=p}^{\infty} n(p+n)|a_n|z^{n+p}}{1 - \sum_{n=p}^{\infty} n|a_n|z^{n+p}}
\]  

(26)

Thus,

\[
1 + z\frac{f''(z)}{f'(z)} + p \leq p - \delta (0 \leq \delta < p)
\]  

(27)

if

\[
\sum_{n=p}^{\infty} n[(p+n) + (p-\delta)]|a_n| |z|^{n+p} \leq 1.
\]  

(28)

Hence, by Theorem 1 (28) will be true if

\[
n[(p+n) + (p-\delta)]|z|^{n+p} \leq \frac{n[1 + \beta(2\varrho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)},
\]  

(29)

the proof of Theorem 4 is completed by merely verifying that each assertion is sharp for the function given by (14). \(\square\)

For functions

\[f_i(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,i}|z^n \quad (i = 1, 2; \ p \in \mathbb{N}),\]

(30)

the Hadamard products (or convolution) of functions \(f_1(z)\) and \(f_2(z)\), is given by

\[(f_1 \ast f_2)(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,1}| |a_{n,2}| z^n.\]

(31)

**Theorem 5.** Let \(f_i(z) \ (i = 1, 2) \in \Lambda^+_{p,q,s}(\alpha_1; \gamma, \varrho, \beta)\), where \(f_i(z) \ (i = 1, 2)\) are in the form (30). Then \((f_1 \ast f_2)(z) \in \Lambda^+_{p,q,s}(\alpha_1; \delta, \varrho, \beta)\), where

\[
\delta = p - \frac{2\beta\varrho(p-\gamma)^2}{p[1 + \beta(2\varrho - 1)]\Gamma_{2p}(\alpha_1)}.
\]  

(32)

**Sharpness holds for functions**

\[f_i(z) = z^{-p} + \frac{2\beta\varrho(p-\gamma)}{p[1 + \beta(2\varrho - 1)]\Gamma_{2p}(\alpha_1)}z^p \quad (i = 1, 2; \ p \in \mathbb{N}).\]

(33)

**Proof.** Using the technique for analytic functions, we need to find the largest real parameter \(\delta\) such that

\[
\sum_{n=p}^{\infty} \frac{n[1 + \beta(2\varrho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)} |a_{n,1}| |a_{n,2}| \leq 1.
\]  

(34)

Since \(f_i(z) \in \Lambda^+_{p,q,s}(\alpha_1; \gamma, \kappa, \beta) \ (i = 1, 2),\) we have

\[
\sum_{n=p}^{\infty} \frac{n[1 + \beta(2\varrho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)} |a_{n,1}| \leq 1
\]  

(35)

and

\[
\sum_{n=p}^{\infty} \frac{n[1 + \beta(2\varrho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p-\gamma)} |a_{n,2}| \leq 1.
\]  

(36)
By Cauchy-Schwarz inequality we have
\[
\sum_{n=p}^{\infty} \frac{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \gamma)} \sqrt{|a_{n,1}| |a_{n,2}|} \leq 1. \tag{37}
\]
thus, it is sufficient to show that
\[
\frac{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \delta)} |a_{n,1}| |a_{n,2}| \leq \frac{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \gamma)} \sqrt{|a_{n,1}| |a_{n,2}|} \tag{38}
\]
or, equivalently, that
\[
\sqrt{|a_{n,1}| |a_{n,2}|} \leq \frac{(p - \delta)}{(p - \gamma)}. \tag{39}
\]
Hence, in the light of the inequality \[(37),\]

it is sufficient to prove that
\[
\frac{2\beta\varrho(p - \gamma)}{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)} \leq \frac{(p - \delta)}{(p - \gamma)}. \tag{40}
\]
It follows from \[(40),\]

that
\[
\delta \leq p - \frac{2\beta\varrho(p - \gamma)^2}{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}. \tag{41}
\]
Let
\[
M(n) = p - \frac{2\beta\varrho(p - \gamma)^2}{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}, \tag{42}
\]
then \(M(n)\) is increasing function of \(n(n \geq p)\). Therefore, we conclude that
\[
\delta \leq M(p) = p - \frac{2\beta\varrho(p - \gamma)^2}{p[1 + \beta(2q - 1)]\Gamma_{2p}(\alpha_1)}. \tag{43}
\]
and hence the proof of Theorem 5 is completed.

**Theorem 6.** Let \(f_1(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, q, \beta)\) and \(f_2(z) \in \Lambda_{p,q,s}^+(\alpha_1; \lambda, q, \beta)\), where \(f_i(z)(i = 1, 2)\) are in the form \[(31).\]

Then \((f_1 * f_2)(z)\) is \(\in \Lambda_{p,q,s}^+(\alpha_1; \xi, q, \beta)\), where
\[
\xi = p - \frac{2\beta\varrho(p - \gamma)(p - \lambda)}{p[1 + \beta(2q - 1)]\Gamma_{2p}(\alpha_1)}. \tag{44}
\]

**Sharpness holds for**
\[
f_1(z) = z^{-p} + \frac{2\beta\varrho(p - \gamma)}{p[1 + \beta(2q - 1)]\Gamma_{2p}(\alpha_1)} z^{\alpha} \tag{45}
\]
and
\[
f_2(z) = z^{-p} + \frac{2\beta\varrho(p - \lambda)}{p[1 + \beta(2q - 1)]\Gamma_{2p}(\alpha_1)} z^{\beta}. \tag{46}
\]

**Proof.** We need to find the largest real parameter \(\xi\) such that
\[
\sum_{n=p}^{\infty} \frac{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \xi)} |a_{n,1}| |a_{n,2}| \leq 1. \tag{47}
\]
Since \(f_1(z) \in \Lambda_{p,q,s}^+(\alpha_1; \gamma, q, \beta)\), we have
\[
\sum_{n=p}^{\infty} \frac{n[1 + \beta(2q - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \gamma)} |a_{n,1}| \leq 1 \tag{48}
\]
and \( f_2(z) \in \Lambda^+_{p,q,s}(\alpha_1; \lambda, \varrho, \beta) \), we have
\[
\sum_{n=p}^{\infty} \frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \lambda)} |a_{n,2}| \leq 1.
\tag{49}
\]

By Cauchy-Schwarz inequality we have
\[
\sum_{n=p}^{\infty} \frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \xi)} |a_{n,1}| |a_{n,2}| \leq \frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \gamma \sqrt{p - \lambda})} \sqrt{|a_{n,1}| |a_{n,2}|}
\tag{50}
\]
thus it is sufficient to show that
\[
\frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \xi)} |a_{n,1}| |a_{n,2}| \leq \frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \gamma \sqrt{p - \lambda})} \sqrt{|a_{n,1}| |a_{n,2}|}
\tag{51}
\]
or, equivalently, that
\[
\sqrt{|a_{n,1}| |a_{n,2}|} \leq \frac{(p - \xi)}{\sqrt{p - \gamma \sqrt{p - \lambda}}}
\tag{52}
\]

Hence, in the light of the inequality \([50]\), it is sufficient to prove that
\[
\frac{2 \beta q(p - \gamma \sqrt{p - \lambda})}{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)} \leq \frac{(p - \xi)}{\sqrt{p - \gamma \sqrt{p - \lambda}}}
\tag{53}
\]
It follows from \([53]\) that
\[
\xi \leq p - \frac{2 \beta q(p - \gamma)(p - \lambda)}{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}
\tag{54}
\]
Let
\[
A(n) = p - \frac{2 \beta q(p - \gamma)(p - \lambda)}{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)},
\tag{55}
\]
then \(A(n)\) is increasing function of \(n(n \geq p)\). Therefore, we conclude that
\[
\xi \leq A(p) = p - \frac{2 \beta q(p - \gamma)(p - \lambda)}{p [1 + \beta (2p - 1)] \Gamma_{2p}(\alpha_1)}
\tag{56}
\]
and hence the proof of Theorem \([6]\) is completed. \(\square\)

**Theorem 7.** Let \( f_i(z)(i = 1, 2) \in \Lambda^+_{p,q,s}(\alpha_1; \alpha, \varphi, \beta) \), where \( f_i(z)(i = 1, 2) \) are in the form \([37]\). Then
\[
h(z) = z^{-p} + \sum_{n=p}^{\infty} (|a_{n,1}|^2 + |a_{n,2}|^2) z^n
\tag{57}
\]
belongs to the class \(\Lambda^+_{p,q,s}(\alpha_1; \varphi, \varrho, \beta)\), where
\[
\varphi = p - \frac{4 \beta q(p - \gamma)^2}{p [1 + \beta (2p - 1)] \Gamma_{2p}(\alpha_1)}.
\tag{58}
\]

Sharpness holds for \( f_i(z)(i = 1, 2) \) defined by \([37]\).

**Proof.** By using Theorem \([5]\) we obtain
\[
\sum_{n=p}^{\infty} \left\{ \frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \gamma)} \right\}^2 |a_{n,1}|^2 \leq \left\{ \sum_{n=p}^{\infty} \frac{n [1 + \beta (2p - 1)] \Gamma_{n+p}(\alpha_1)}{2 \beta q(p - \gamma)} |a_{n,1}| \right\}^2 \leq 1,
\tag{59}
\]
and
\[ \sum_{n=p}^\infty \left\{ \frac{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \gamma)} \right\} |a_{n,2}|^2 \leq \left( \sum_{n=p}^\infty \frac{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \gamma)} |a_{n,2}| \right)^2 \leq 1. \] (60)

It follows from (59) and (60) that
\[ \sum_{n=p}^\infty \frac{1}{8} \left\{ \frac{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}{\beta\varrho(p - \gamma)} \right\}^2 (|a_{n,1}|^2 + |a_{n,2}|^2) \leq 1. \] (61)

Therefore, we need to find the largest \( \varphi \) such that
\[ \frac{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}{2\beta\varrho(p - \varphi)} \leq \frac{1}{8} \left\{ \frac{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}{\beta\varrho(p - \gamma)} \right\}^2 \] (62)

that is
\[ \varphi \leq p - \frac{4\beta\varrho(p - \gamma)^2}{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}. \] (63)

Let
\[ H(n) = p - \frac{4\beta\varrho(p - \gamma)^2}{n[1 + \beta(2\rho - 1)]\Gamma_{n+p}(\alpha_1)}, \] (64)

then \( H(n) \) is an increasing function of \( n(n \geq p) \). Therefore, we conclude that
\[ \varphi \leq H(p) = p - \frac{4\beta\varrho(p - \gamma)^2}{p[1 + \beta(2\rho - 1)]\Gamma_{2p}(\alpha_1)} \] (65)

and hence the proof of Theorem 7 is completed. \( \square \)

**Remark 8.** Putting \( \rho = 1 \) in our results we obtain the results obtained by Aouf [3] with \( A = 1 \) and \( B = -1 \).

**References**
