BEHAVIOR OF SOLUTIONS OF A CLASS OF NONLINEAR RATIONAL DIFFERENCE EQUATION $x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}}{\gamma x_{n-s}}$

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Abstract. The main objective of this paper is to study the qualitative behavior for a class of nonlinear rational difference equation. We study the local stability, periodicity, oscillation, boundedness, and the global stability for the positive solutions of equation. Examples illustrate the importance of the results.

1. Introduction

In this paper, we aim to achieve a qualitative study of some behavior and solutions in a non-linear difference equations

$$x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}}{\gamma x_{n-s}}, \quad n = 0, 1, 2, ..., \tag{1}$$

where the coefficients $\alpha, \beta$ and $\gamma \in (0, \infty)$ while $k, \ell$ and $s$ are positive integers. The initial conditions $x_{-j}, x_{-j+1}, ..., x_0$ are arbitrary positive real numbers such that $j = -\max \{k, \ell, s\}$. Consider $\delta \in [1, \infty)$. Qualitative analysis of difference equation is not only interesting in its own right, but it can provide insights into their continuous counterparts, namely, differential equations.

There is a set of nonlinear difference equations, known as the rational difference equations, all of which consists of the ratio of two polynomials in the sequence terms in the same from. There has been much work about the global asymptotic of solutions of rational difference equations \cite{3, 4, 7, 8, 11, 12}.

In the following we present some basic definitions and known results which will be useful in our study.

Definition 1. Consider a difference equation in the form

$$x_{n+1} = F(x_{n-k}, x_{n-\ell}, x_{n-s}) \tag{2}$$

where $F$ is a continuous function, while $k, \ell, s \in (0, \infty)$ are positive integers. An equilibrium point $\overline{x}$ of this equation is a point that satisfies the condition $\overline{x} = \ldots$
$F(x, x, x)$. That is, the constant sequence \( \{x_n\} \) with \( x_n = x \) for all \( n \geq -k \geq -\ell \) is a solution of that equation.

**Definition 2.** [3] Let \(x \in (0, \infty)\) be an equilibrium point of Eq. [2]. Then we have

(i) An equilibrium point \(x\) of Eq. is said to be locally stable if for every \(\varepsilon > 0\) there exists \(\sigma > 0\) such that, if \(x_{-j}, \ldots, x_{-1}, x_0 \in (0, \infty)\) with \(|x_{-j} - x| + \ldots + |x_{-1} - x| + |x_0 - x| < \sigma\), then \(|x_n - x| < \varepsilon\) for all \(n \geq -j\).

(ii) An equilibrium point \(x\) of Eq. [2] is said to be locally asymptotically stable if it is locally stable and there exists \(y > 0\) such that, \(x_{-j}, \ldots, x_{-1}, x_0 \in (0, \infty)\) with \(|x_{-j} - x| + \ldots + |x_{-1} - x| + |x_0 - x| < y\), then \(\lim_{n \to \infty} x_n = x\).

(iii) An equilibrium point \(x\) of Eq. [2] is said to be globally asymptotically stable if it is locally stable and a global attractor.

(iv) An equilibrium point \(x\) of Eq. [2] is said to be unstable if it is not locally stable.

(v) An equilibrium point \(x\) of Eq. [2] is said to be unstable if it is not locally stable.

**Definition 3.** [3] The sequence \(\{x_n\}\) is said to be periodic with period \(p\) if \(x_{n+p} = x_n\) for \(n = 0, 1, \ldots\).

**Definition 4.** [3] Eq. [3] is said to be permanent and bounded if there exists numbers \(m\) and \(M\) with \(0 < m < M < \infty\) such that for any initial conditions \(x_{-j}, \ldots, x_{-1}, x_0 \in (0, \infty)\) there exists a positive integer \(N\) which depends on these initial conditions such that \(m \leq x_n \leq M\) for all \(n \geq N\).

**Definition 5.** [3] A sequence \(\{x_n\}_{n=-k}^{\infty}\) is said to be nonoscillatory about the point \(x\) if there is exists \(N \geq -k\) such that either \(x_n > x\) for all \(n \geq N\) or \(x_n < x\) for all \(n \geq N\). Otherwise \(\{x_n\}_{n=-k}^{\infty}\) is called oscillatory about \(x\).

**Definition 6.** [3] The linearized equation of Eq. [3] about the equilibrium point \(x\) is defined by the equation.

\[
y_{n+1} = p_0y_{n-k} + p_1y_{n-\ell} + p_2y_{n-s}
\]

\[
p_0 = \frac{\partial f}{\partial x_{n-k}}(x, x, x), \quad p_1 = \frac{\partial f}{\partial x_{n-\ell}}(x, x, x), \quad p_2 = \frac{\partial f}{\partial x_{n-s}}(x, x, x)
\]

The characteristic equation associated with Eq. [3] is

\[
p(\lambda) = \lambda^{n+1} - p_0\lambda^n - p_1\lambda^\ell - p_2 = 0
\]

**Theorem 1.** [3] Assume that \(p_0, p_1\) and \(p_2 \in \mathbb{R}\). Then

\[
|p_0| + |p_1| + |p_2| < 1
\]

is a sufficient condition for the locally stability of Eq. [2].

2. Local stable of the equilibrium point

The equilibrium point of Eq. [1] is the positive solution of the equation

\[
\bar{x} = \alpha \bar{x} + \frac{\alpha \bar{x}^\ell}{b \bar{x}^3}
\]

which gives

\[
x = \frac{\beta}{\gamma(1 - \alpha)}, \quad \alpha < 1
\]
Now let \( f : (0, \infty)^3 \to (0, \infty) \) be a function defined by
\[
f(u, v, w) = \alpha u + \frac{\beta v^\delta}{\gamma w^\delta}.
\]

Then, we have
\[
\frac{\partial f}{\partial u} = \alpha, \quad \frac{\partial f}{\partial v} = \frac{\beta \delta v^{\delta-1}}{\gamma w^\delta}, \quad \frac{\partial f}{\partial w} = \frac{-\beta \gamma \delta v^\delta w^{\delta-1}}{(\gamma w^\delta)^2}.
\]

**Theorem 2.** If
\[
\alpha + 2\delta < 1 + 2\alpha \delta
\]
then the equilibrium point \( \bar{x} \) of eq (1) is locally stable.

**Proof.** From (7) to (9), we get
\[
\frac{\partial f}{\partial u} (\bar{x}, \bar{x}, \bar{x}) = \alpha = p_0,
\]
\[
\frac{\partial f}{\partial v} (\bar{x}, \bar{x}, \bar{x}) = \delta (1 - \alpha) = p_1,
\]
and
\[
\frac{\partial f}{\partial w} (\bar{x}, \bar{x}, \bar{x}) = -\delta (1 - \alpha) = p_2.
\]

Thus, the linearized equation associated with Eq. (2) about \( \bar{x} \), is
\[
y_{n+1} = p_0 y_{n-k} + p_1 y_{n-\ell} + p_2 y_{n-s}.
\]
It follows by Theorem 1 that Eq. (1) is locally stable if
\[
|\alpha| + |\delta (1 - \alpha)| + |\delta (1 - \alpha)| < 1,
\]
after simplification and calculations, we get
\[
\alpha + 2\delta (1 - \alpha) < 1,
\]
which is true if
\[
\alpha + 2\delta < 1 + 2\alpha \delta.
\]
The proof is completed. \( \square \)
Example 1. Fig. 1, shows that Eq. (1) has Local stable solutions if \( \alpha = 0.5, \beta = \gamma = 1, k = \ell = s = \delta = 1, x_0 = 1.5, x_{-1} = 5.4, x_{-2} = 1.3, \tau = 2. \)

In this part of the research we are studying the possibility of the existence of periodic solutions to the eq. (1).

Theorem 3. If \( \delta = 1. \) In the all following cases, Equation (1) has no positive prime period-two solutions:

1. If \( k, \ell \) and \( s \) are all even positive number.
2. If \( k, \ell \) and \( s \) are all odd positive number.
3. If \( k \) is even and \( \ell, s \) are both odd positive number.
4. If \( k, \ell \) are both even and \( s \) odd positive number.
5. If \( k \) is odd and \( \ell, s \) are both even positive number.
6. If \( k, \ell \) are both odd and \( s \) is even positive number.
7. If \( k, s \) are both odd and \( \ell \) is even positive number.

Proof. Case(1) Suppose that there exists a prime period-two solution

\[ ..., p, q, p, q, p, q, ..., \]

If \( k, \ell \) even then \( x_n = x_{n-k} = x_{n-\ell} = x_{n-s} = q, x_{n+1} = p \)

\[ p = \alpha q + \frac{\beta}{\gamma}, \quad (10) \]

also,

\[ q = \alpha p + \frac{\beta}{\gamma}. \quad (11) \]

By (10) and (11), we have

\[ (p - q)(\alpha + 1) = 0 \quad \implies \quad p = q \]
Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed.

The following theorem states the sufficient conditions that the Eq. (1) has periodic solutions of prime period two.

**Theorem 4.** Assume that \( k, s \) are both even and \( \ell \) is odd positive integers and \( \delta = 1 \). If

\[
3\alpha < 1,
\]

then Eq. (1) has prime period two solution.

**Proof.** Suppose that there exists a prime period-two solution

\[
\ldots, p, q, p, q, p, q, \ldots
\]

of (1). We will prove that condition (12) holds.

We see from (1) that if \( k, s \) are both even and \( \ell \) is odd, then

\[
x_n = x_{n-k} = x_{n-s} = q, \quad x_{n+1} = x_{n-\ell} = p
\]

we have

\[
\gamma pq = \alpha \gamma q^2 + \beta p, \tag{13}
\]

and

\[
\gamma pq = \alpha \gamma p^2 + \beta q. \tag{14}
\]

By subtracting (13) and (14), we have

\[
\alpha \gamma (q^2 - p^2) + \beta (p - q) = 0,
\]

then,

\[
(p + q) = \frac{\beta}{\alpha \gamma}. \tag{15}
\]

By Combining (13) and (14), we have

\[
2\gamma pq = \alpha \gamma (p^2 + q^2) + \beta (p + q), \tag{16}
\]

then,

\[
p^2 + q^2 = (p + q)^2 - 2pq. \tag{17}
\]

Form (15), (16) and (17), we get

\[
2\gamma pq (1 + \alpha) = \alpha \gamma \left[ \frac{\beta}{\alpha \gamma} \right]^2 + \beta \left[ \frac{\beta}{\alpha \gamma} \right],
\]

\[
pq = \frac{\beta^2}{\alpha \gamma^2 (1 + \alpha)}.
\]

We have,

\[
u^2 + (p + q) u + pq = 0 \quad \text{and} \quad (p + q)^2 - 4pq > 0,
\]

then,

\[
\left[ \frac{\beta}{\alpha \gamma} \right]^2 - \frac{4\beta^2}{\alpha \gamma^2 (1 + \alpha)} > 0,
\]

which is true if

\[
3\alpha < 1.
\]
Hence, the proof is completed.

Example 2. Fig. 2, shows that Eq. (7) has prime period two solutions if \( k = s = 0, \ell = 1, \alpha = (1/16), \beta = 2, \gamma = \delta = 1 \), (see Table 1)

![Graph showing oscillatory behavior]

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Table 1

4. Global stability

Theorem 5. If \( \alpha < 1 \), then the equilibrium point \( \xi \) of Eq. (7) is global attractor.
Proof. We consider the following function
\[ f(u, v, w) = \alpha u + \frac{\beta v^\delta}{\gamma w^\sigma}, \]
are increasing for \( u, v \) and decreasing for \( w \).
Let \( m = f(m, m, M) \) and \( M = f(M, M, m) \)
\[ m = \alpha m + \frac{\beta m^\delta}{\gamma M^\delta}, \quad \tag{18} \]
\[ M = \alpha M + \frac{\beta M^\delta}{\gamma m^\sigma}, \quad \tag{19} \]
from (18)
\[ \gamma m M^\delta (1 - \alpha) = \beta m^\delta, \quad \tag{20} \]
from (19)
\[ \gamma M m^\delta (1 - \alpha) = \beta M^\delta. \quad \tag{21} \]
Subtracting Equation (20) of (21) produces
\[ \gamma (1 - \alpha) (mM^\delta - Mm^\delta) - \beta (m^\delta - M^\delta) = 0, \]
then
\[ M = m \]
Hence, the proof is completed. \( \square \)

5. Oscillatory solution

Theorem 6. Eq. (1) has an oscilatory solution If \( k = \max \{k, \ell, s\} \) and \( k, \ell \) is odd and \( s \) is even.

Proof. First, assume that,
\[ x_{-k}, x_{-k+2}, x_{-k+4}, \ldots, x_{-1} > \overline{x} \quad \text{and} \quad x_{-k+1}, x_{-k+3}, \ldots, x_0 < \overline{x} \]
so
\[ x_1 = \alpha x_{-k} + \frac{\beta x_{-\ell}}{\gamma x_{-s}}, \]
then
\[ x_1 > \alpha \overline{x} + \frac{\beta \overline{x}^\gamma}{\gamma \overline{x}^\sigma}, \]
and
\[ x_1 > \frac{\beta}{\gamma (1 - \alpha)} = \overline{x}. \]
So, we have
\[ x_2 = \alpha x_{-k+1} + \frac{\beta x_{-\ell+1}}{\gamma x_{-s+1}}, \]
so,
\[ x_2 < \alpha + \frac{\alpha \overline{x}^\gamma}{\beta \overline{x}^\gamma + \overline{x}^\sigma}, \]
then,
\[ x_2 < \frac{\beta}{\gamma (1 - \alpha)} = \overline{x}. \]
Secondly assume that,
\[ x_{-k}, x_{-k+2}, x_{-k+4}, \ldots, x_{-1} < \overline{x} \quad \text{and} \quad x_{-k+1}, x_{-k+3}, \ldots, x_0 > \overline{x}, \]
\[ x_1 = \alpha x_{-k} + \frac{\beta x_{-t}}{\gamma x_{-s}} \]
then,
\[ x_1 < \alpha \bar{x} + \frac{\beta \bar{x}^\gamma}{\gamma \bar{x}^s}, \]
and
\[ x_1 < \frac{\beta}{\gamma (1 - \alpha)} = \bar{x}. \]
So, we have
\[ x_2 = \alpha x_{-k+1} + \frac{\beta x_{-t+1}}{\gamma x_{-s+1}}, \]
so,
\[ x_2 > \alpha \bar{x} + \frac{\beta \bar{x}^\gamma}{\gamma \bar{x}^s}, \]
then,
\[ x_2 > \frac{\beta}{\gamma (1 - \alpha)} = \bar{x}. \]
One can proceed in prove manuer to show that \( x_3 < \bar{x} \) and \( x_4 > \bar{x} \) and soon. Hence, the proof is completed. \( \Box \)

**Example 3.** Fig. 3, shows that Eq.(1) has oscilatory solution if \( \alpha = 0.5, \beta = 5, \gamma = 5, \delta = 0.5, \bar{x} = 2. \) (see Table 2)
Table 2

6. Boundedness of the solutions

**Theorem 7.** Let \( \{x_n\}_{n=-\infty}^{\infty} \) be a solution of Eq. (1), then the following statements are true:

1. Assume that \( \beta < \gamma \) and let for some \( N \geq 0, x_{N-\ell+1}, \ldots, x_N \in \left[ \frac{\beta}{\gamma}, 1 \right] \) are valid, then we have

\[
\frac{\alpha \beta^\delta}{\gamma^\delta} + \frac{\beta^\delta}{\gamma^\delta - 1} \leq x_n \leq \frac{\gamma^\delta - 1}{\beta^\delta - 1}
\]

2. Assume that \( \beta > \gamma \) and for some \( N \geq 0, x_{N-\ell+1}, \ldots, x_N \in \left[ 1, \frac{\beta}{\gamma} \right] \) are valid, then we have

\[
\frac{\alpha + \gamma^\delta - 1}{\beta^\delta - 1} \leq x_n \leq \frac{\alpha \beta^\delta}{\gamma^\delta} + \frac{\beta^\delta}{\gamma^\delta - 1}
\]

**Proof.** (1) If \( \beta < \gamma \) then \( x_{N-\ell+1}, \ldots, x_{N-1}, x_N \in \left[ \frac{\beta}{\gamma}, 1 \right] \)

\[
x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}}{\gamma x_{n-s}}
\]

then,

\[
\leq \alpha + \frac{\beta}{\gamma \left( \frac{\beta}{\gamma} \right)},
\]

\[
\leq \alpha + \frac{\gamma^\delta - 1}{\beta^\delta - 1},
\]

and

\[
x_{n+1} = \alpha x_{n-k} + \frac{\beta x_{n-\ell}}{\gamma x_{n-s}}.
\]

then,
\[ \geq \frac{\alpha \beta}{\gamma} + \frac{\beta^{\delta+1}}{\gamma^{\delta-1}}. \]

Then
\[ \frac{\alpha \beta}{\gamma} + \frac{\beta^{\delta+1}}{\gamma^{\delta-1}} \leq x_n \leq \alpha + \frac{\sqrt{\delta-1}}{\beta^{\delta-1}}. \]

Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed.

\[ \square \]

References


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