SOLVABILITY OF DEGENERATED $p(x)$-PARABOLIC EQUATIONS WITH THREE UNBOUNDED NONLINEARITIES.

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Abstract. In this paper, we study the existence of renormalized solutions for the nonlinear $p(x)$-parabolic problem with $f \in L^1(Q)$ and $b(x,u_0) \in L^1(\Omega)$. The main contribution of our work is to prove the existence of renormalized solutions of the weighted variable exponent Sobolev spaces and we suppose that $H(x,t,u,\nabla u)$ is the nonlinear term satisfying some growth condition but no sign condition or the coercivity condition.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N \geq 1)$, $T$ is a positive real number, and $Q = \Omega \times (0,T)$. We are interested in existence of renormalized solutions to the following nonlinear parabolic problem

\[
\begin{aligned}
\frac{\partial b(x,u)}{\partial t} - \text{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) &= f \text{ in } Q = \Omega \times (0,T) \\
b(x,u) \big|_{t=0} &= b(x,u_0) \text{ in } \Omega \\
0 &= \text{ on } \partial \Omega \times (0,T),
\end{aligned}
\]

where $f \in L^1(Q)$, $b(x,u_0) \in L^1(\Omega)$. The operator $-\text{div}(a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on $L^{p(\cdot)}(\Omega)$ (see assumption (3.3)-(3.5) of section 3) which is coercive $b(x,u)$ is an unbounded function of $u$, $H$ is a nonlinear lower order term. The notion of renormalized solutions was introduced by R. J. Diperna and P. L. Lions for the study of the Boltzmann equation. It was then used by L. Boccardo and al when the right hand side is in $W^{-1,p'(\Omega)}$ and by J. M Rakoston when the right hand side is in $L^1(\Omega)$. It is our purpose to prove the existence of renormalized solution of weighted variable exponent Sobolev spaces for the problem $(P)$ setting without the sign condition and without the coercivity condition, the critical growth condition on $H$ is only with respect to $\nabla u$ and not with respect to $u$ (see assumption H2). Where the right hand side is assumed to satisfy: $f$ belongs to $L^1(Q)$. Other work in this direction can be found in [1, 4, 19, 20].

For the convenience of the readers, we recall some definitions and basic properties of


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the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega, \omega)$ and the weighted variable exponent Sobolev spaces $W^{1, p(x)}(\Omega, \omega)$. Set

$$C_+ (\underline{p}) = \{ p \in C(\underline{p}) : \min_{x \in \Omega} p(x) > 1 \}.$$ 

For any $p \in C_+ (\underline{p})$, we define $p^+ = \max_{x \in \Omega} p(x)$, $p^- = \min_{x \in \Omega} p(x)$. For any $p \in C_+ (\underline{p})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega, \omega)$ that consists of all measurable real-valued functions $u$ such that

$$L^{p(x)}(\Omega, \omega) = \{ u : \Omega \to \mathbb{R}, \text{measurable}, \int_{\Omega} |u(x)|^{p(x)} \omega(x) \, dx < \infty \}.$$ 

Then, $L^{p(x)}(\Omega, \omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega, \omega)} = \inf \{ \lambda > 0 : \int_{\Omega} \frac{u(x)}{\lambda} |p(x)\omega(x) \, dx \leq 1 \}$$

becomes a normed space. When $\omega(x) \equiv 1$, we have $L^{p(x)}(\Omega, \omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega, \omega)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1, p(x)}(\Omega, \omega)$ is defined by

$$W^{1, p(x)}(\Omega, \omega) = \{ u \in L^{p(x)}(\Omega) ; |\nabla u| \in L^{p(x)}(\Omega, \omega) \},$$

where the norm is

$$\| u \|_{W^{1, p(x)}(\Omega, \omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega, \omega)} \quad (1.1)$$

or, equivalently

$$\| u \|_{W^{1, p(x)}(\Omega, \omega)} = \inf \{ \lambda > 0 : \int_{\Omega} \frac{u(x)}{\lambda} |p(x)+\omega(x)| \nabla u(x) |p(x) \, dx \leq 1 \}$$

for all $u \in W^{1, p(x)}(\Omega, \omega)$.

It is significant that smooth functions are not dense in $W^{1, p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [21] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant $C$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log |x - y|} \quad (1.2)$$

for every $x, y$ with $|x - y| \leq \frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W^{1, p(x)}(\Omega)$, as the completion of $C_0^\infty (\Omega)$ with respect to the norm $\| u \|_{W^{1, p(x)}(\Omega)}$ (see [12]).

$W^{1, p(x)}_0 (\Omega, \omega)$ is defined as the completion of $C_0^\infty (\Omega)$ in $W^{1, p(x)}(\Omega, \omega)$ with respect to the norm $\| u \|_{W^{1, p(x)}(\Omega, \omega)}$.

Throughout the paper, we assume that $p \in C_+ (\underline{p})$ and $\omega$ is a measurable positive and a.e. finite function in $\Omega$.

This paper is organized as follows. In Section 2, we state some basic results for the weighted variable exponent Lebesgue-Sobolev spaces which is given in [16]. In Section 3, we make precise all the assumption on $b, a, H, f$ and $b(x, u_0)$ and give the definition of a renormalized solution of the problem $(P)$ and main results, which is proved in Section 4.

2. Preliminaries

In this Section, we state some elementary properties for the (weighted) variable exponent Lebesgue-Sobolev spaces which will be used in the next sections. The basic properties of the variable exponent Lebesgue-Sobolev spaces, that is when $\omega(x) \equiv 1$ can be found from [13] [15].
Remark 2.4. (Generalised Hölder inequality).

i) For any functions $u \in L^{p_1}(\Omega)$ and $v \in L^{p_2}(\Omega)$, we have
\[ |\int \frac{u}{v} | dx \leq \left( \frac{1}{p_1} + \frac{1}{p_2} \right) \| u \|_{p_1} \| v \|_{p_2} \leq 2 \| u \|_{p_1} \| v \|_{p_2}. \]

ii) Moreover, if $\omega$ is a positive measurable and finite function, then
\[ \frac{u}{\omega} \in L^{p}(\Omega) \text{ if and only if } \| u \|_{p} \leq \omega \| u \|_{p}. \]

Lemma 2.2. (See [16].) Denote $\rho(u) = \int_{\Omega} \omega(x)|u(x)|^{p(x)} dx$ for all $u \in L^{p(x)}(\Omega, \omega)$.

Then,
\[ |u|_{L^{p(x)}(\Omega, \omega)} < 1 (= 1; > 1) \text{ if and only if } \rho(u) < 1 (= 1; > 1), \]
\[ \text{if } |u|_{L^{p(x)}(\Omega, \omega)} > 1 \text{ then } |u|_{L^{p(x)}(\Omega, \omega)} \leq \rho(u) \leq |u|_{L^{p(x)}(\Omega, \omega)^+}, \]
\[ \text{if } |u|_{L^{p(x)}(\Omega, \omega)} < 1 \text{ then } |u|_{L^{p(x)}(\Omega, \omega)} \leq \rho(u) \leq |u|_{L^{p(x)}(\Omega, \omega)^-}. \]

Remark 2.3. (See [17].) If we set
\[ I(u) = \int_{\Omega} |u(x)|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)} dx. \]

Then, following the same argument, we have
\[ \min \{ \| u \|_{W^{1,p(x)}(\Omega, \omega)}, \| u \|_{W^{1,p(x)}(\Omega, \omega)^+} \} \leq I(u) \leq \max \{ \| u \|_{W^{1,p(x)}(\Omega, \omega)^-}, \| u \|_{W^{1,p(x)}(\Omega, \omega)^+} \}. \]

Throughout the paper, we assume that $\omega$ is a measurable and positive function in $\Omega$ satisfying that
- $\omega \in L^{\frac{1}{p(x)-1}}(\Omega)$;
- $\omega^{-\frac{1}{p(x)-1}} \in L_{loc}^{\frac{1}{p(x)-1}}(\Omega)$;
- $\omega^{-s(x)} \in L^{1}(\Omega)$ with $s(x) \in (\frac{N}{p(x)} - \infty, \infty)$.

The reasons that we assume (W1) and (W2) can be found in [16].

Remark 2.4. (See [17].)

(i) If $\omega$ is a positive measurable and finite function, then $L^{p(x)}(\Omega, \omega)$ is a reflexive Banach space.

(ii) Moreover, if (W1) holds, then $W^{1,p(x)}(\Omega, \omega)$ is a reflexive Banach space.

For $p, s \in C_{+}(\overline{\Omega})$, denote
\[ p_s(x) = \frac{p(x)}{s(x) + 1} < p(x), \]
where $s(x)$ is given in (W2).

Assume that we fix the variable exponent restrictions
\[ \begin{cases} p_s(x) = \frac{p(x)^{s(x)} N}{s(x) + 1} & \text{if } N > p_s(x), \\ p_s(x) \text{ arbitrary} & \text{if } N \leq p_s(x) \end{cases} \]
for almost all $x \in \Omega$. These definitions play a key role in our paper. We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue-Sobolev space in the next sections.

Lemma 2.5. (See [16].) Let $p, s \in C_{+}(\overline{\Omega})$ satisfy the log-Hölder continuity condition [1.2], and let (W1) and (W2) be satisfied. If $r \in C_{+}(\overline{\Omega})$ and $1 < r(x) \leq p_s$, then, we obtain the continuous imbedding
\[ W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega), \]
Moreover, we have the compact imbedding
\[ W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega), \]
provided that $1 < r(x) < p_s(x)$ for all $x \in \overline{\Omega}$.

From Lemma 2.5, we have Poincaré-type inequality immediately.
Corollary 2.6. (15) Let $p \in C_{+}(\Omega)$ satisfy the log-Hölder continuity condition (1.2). If (W1) and (W2) hold, then the estimate
$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega, \omega)}$$
holds, for every $u \in C^{0}_{\alpha}(\Omega)$ with a positive constant $C$ independent of $u$.

Throughout this paper, let $p \in C_{+}(\Omega)$ satisfy the log-Hölder continuity condition (1.2) and $X := W^{1,p(x)}(\Omega, \omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions $u$ from $W^{1,p(x)}(\Omega, \omega)$ which vanish on the boundary $\partial \Omega$, endowed with the norm
$$\|u\|_{X} = \inf \{\lambda > 0 : \int_{\Omega} \left| \nabla u(x) \right|^{p(x)} \omega(x) dx \leq 1\},$$
which is equivalent to the norm (1.1) due to Corollary 2.6. The following proposition gives the characterization of the dual space $(W^{1,p(x)}_{0}(\Omega, \omega))^{\ast}$, which is analogous to [15, Theorem 3.16]. We recall that the dual space of weighted Sobolev spaces $W^{1,p(x)}_{0}(\Omega, \omega)$ is equivalent to $W^{-1,p(x)}(\Omega, \omega)$, where $\omega^{\ast} = \omega^{1-p(x)}$.

Lemma 2.7. (15) Let $g \in L^{p(\cdot)}(Q, \omega)$ and let $g_{n} \in L^{p(\cdot)}(Q, \omega)$, with $\|g_{n}\|_{L^{p(\cdot)}(Q, \omega)} \leq c$, $1 < r(x) < \infty$. If $g_{n} \rightharpoonup g$ a.e. in $Q$, then $g_{n} \rightharpoonup g$ in $L^{p(\cdot)}(Q, \omega)$, where $\rightharpoonup$ denotes weak convergence and $\omega$ is a weight function on $Q$.

We will also use the standard notation for Bochner spaces, i.e., if $g \geq 1$ and $X$ is a Banach space then $L^{g}(0, T; X)$ denotes the space of strongly measurable function $u : (0, T) \to X$ for which $t \to \|u(t)\|_{X} \in L^{g}(0, T)$. Moreover, $C([0, T]; X)$ denotes the space of continuous function $u : [0, T] \to X$ endowed with the norm $\|u\|_{C([0, T]; X)} = \max_{t \in [0, T]} \|u(t)\|_{X}$.

Let
$$L^{p(\cdot)}_{-}(0, T; W^{1,p(\cdot)}_{0}(\Omega, \omega)) = \{u : (0, T) \to W^{1,p(\cdot)}_{0}(\Omega, \omega) \text{ measurable;}
\left(\int_{0}^{T} \|u(t)\|_{W^{1,p(\cdot)}_{0}(\Omega, \omega)}^{p(\cdot)}ight)^{1/p(\cdot)} < \infty\}$$
and we define the space
$$L^{\infty}(0, T; X) = \{u : (0, T) \to X \text{ measurable; } \exists C > 0 / \|u(t)\|_{X} \leq C \text{ a.e.}\}$$
where the norm is defined by:
$$\|u\|_{L^{\infty}(0, T; X)} = \inf \{C > 0 / \|u(t)\|_{X} \leq C \text{ a.e.}\}.$$ We introduce the functional space see [5]
$$V = \{f \in L^{p(\cdot)}_{-}(0, T; W^{1,p(\cdot)}_{0}(\Omega, \omega)); \|\nabla f\| \in L^{p(\cdot)}(Q, \omega)\},$$
which endowed with the norm:
$$\|f\|_{V} = \|\nabla f\|_{L^{p(\cdot)}(Q, \omega)}$$
or, the equivalent norm :
$$\|f\|_{V} = \|f\|_{L^{p(\cdot)}_{-}(0, T; W^{1,p(\cdot)}_{0}(\Omega, \omega))} + \|\nabla f\|_{L^{p(\cdot)}(Q, \omega)},$$
is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(\cdot)}_{-}(Q) \hookrightarrow L^{p(\cdot)}_{-}(0, T; L^{p(\cdot)}(\Omega))$ and the Poincaré inequality. We state some further properties of $V$ in the following lemma.

Lemma 2.8. Let $V$ be defined as in (2.4) and its dual space be denote by $V^{\ast}$. Then, i) We have the following continuous dense embeddings:
$$L^{p(\cdot)}_{-}(0, T; W^{1,p(\cdot)}_{0}(\Omega, \omega)) \hookrightarrow V \hookrightarrow L^{p(\cdot)}_{-}(0, T; W^{1,p(\cdot)}_{0}(\Omega, \omega)).$$
In particular, since $D(Q)$ is dense in $L^{p^*}(0,T; W_0^{1,p^*}(\Omega,\omega))$, it is dense in $V$ and for the corresponding dual spaces, we have

$$L^{p^*}(0,T; (W_0^{1,p^*}(\Omega,\omega))^*) \hookrightarrow V^* \hookrightarrow L^{p^*}(0,T; W_0^{1,p^*}(\Omega,\omega))^*.$$ 

Note that, we have the following continuous dense embeddings

$$L^{p^*}(0,T; L^{p^*}(\Omega,\omega)) \hookrightarrow L^{p^*}(\Omega,\omega) \hookrightarrow L^{p^*}(0,T; L^{p^*}(\Omega,\omega)).$$

ii) One can represent the elements of $V^*$ as follows: if $T \in V^*$, then there exists $F = (f_1, ..., f_N) \in (L^p(\Omega))^N$ such that $T = \text{div}_X F$ and

$$\langle T, \xi \rangle_{V^*, V} = \int_0^T \int_{\Omega} F \cdot \nabla \xi dx dt$$

for any $\xi \in V$. Moreover, we have

$$\|T\|_{V^*} = \max\{\|f_i\|_{L^p(\Omega,\omega)}, i = 1, ..., n\}.$$

Remark 2.9. The space $V \cap L^\infty(Q)$, is endowed with the norm defined by the formula:

$$\|v\|_{V \cap L^\infty(Q)} = \max\{\|v\|_V, \|v\|_{L^\infty(Q)}\}, v \in V \cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V + L^1(Q)$ endowed with the norm:

$$\|v\|_{V + L^1(Q)} := \inf\{\|v_1\|_V + \|v_2\|_{L^1(Q)}; v = v_1 + v_2, v_1 \in V, v_2 \in L^1(Q)\}.$$ 

2.1. Some Technical Results.

Lemma 2.10. Assume (3.3) - (3.5) and let $(u_n)_n$ be a sequence in $L^{p^*}(0,T; W_0^{1,p^*}(\Omega,\omega))$ such that $u_n \rightharpoonup u$ weakly in $L^{p^*}(0,T; W_0^{1,p^*}(\Omega,\omega))$ and

$$\int_Q \left( a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u) \right) \cdot \nabla (u_n - u) dx dt \to 0.$$ 

Then, $u_n \to u$ strongly in $L^{p^*}(0,T; W_0^{1,p^*}(\Omega,\omega))$.

Proof.

Let $D_n = [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)] \nabla (u_n - u)$, thanks to (3.3), we have $D_n$ is a positive function, and by (2.5), $D_n \to 0$ in $L^1(Q)$ as $n \to \infty$.

Extracting a subsequence, still denoted by $u_n$, we can write $u_n \to u$ a.e. in $Q$ and since $D_n \to 0$ a.e. in $Q$. There exists a subset $B$ in $Q$ with measure zero such that for all $(t, x) \in Q \setminus B$,

$$|u(x,t)| < \infty, \quad |\nabla u(x,t)| < \infty, \quad K(x,t) < \infty, \quad u_n \to u, \quad D_n \to 0.$$

Taking $\xi_n = \nabla u_n$ and $\xi = \nabla u$, we have

$$D_n(x,t) = [a(x,t,u_n,\xi_n) - a(x,t,u_n,\xi)] \cdot (\xi_n - \xi)
\geq \alpha \omega(x)|\xi_n|^{p(x)} + \alpha \omega(x)|\xi|^{p(x)}
- \beta \omega^{1/p(x)}(x) \left( k(x,t) + \omega^{1/p(x)}(x)|u_n|^{p(x)-1} + \omega^{1/p(x)}(x)|\xi_n|^{p(x)-1} \right) |\xi_n|
- \beta \omega^{1/p(x)}(x) \left( k(x,t) + \omega^{1/p(x)}(x)|u_n|^{p(x)-1} + \omega^{1/p(x)}(x)|\xi|^{p(x)-1} \right) |\xi|,
\geq \alpha \omega(x)|\xi_n|^{p(x)} - C_{x,t}[1 + \omega^{1/p(x)}(x)|\xi_n|^{p(x)-1} + \omega^{1/p(x)}(x)|\xi|^{p(x)-1}],$$

where $C_{x,t}$ depending on $x$, but does not depend on $n$. (Since $u_n(x,t) \to u(x,t)$ then, $(u_n)_n$ is bounded), we obtain

$$D_n(x,t) \geq |\xi_n|^{p(x)} \left( \alpha \omega(x) - \frac{C_{x,t}\omega^{1/p(x)}}{|\xi_n|^{p(x)-1}} - \frac{C_{x,t}\omega^{1/p(x)}}{|\xi_n|^{p(x)-1}} \right),$$

where $C_{x,t}$ depending on $x$, but does not depend on $n$. (Since $u_n(x,t) \to u(x,t)$ then, $(u_n)_n$ is bounded), we obtain

$$D_n(x,t) \geq |\xi_n|^{p(x)} \left( \frac{\alpha \omega(x)}{|\xi_n|} - \frac{C_{x,t}\omega^{1/p(x)}}{|\xi_n|^{p(x)-1}} - \frac{C_{x,t}\omega^{1/p(x)}}{|\xi_n|^{p(x)-1}} \right).$$
Let \(\xi_n\) is bounded almost everywhere in \(Q\). Indeed, if \(|\xi_n| \to \infty\) in a measurable subset \(E \subset Q\), then

\[
\lim_{n \to \infty} \int_Q D_n(x,t)dx \geq \lim_{n \to \infty} \int_E |\xi_n|^{p(x)}\left(\alpha_1 + \frac{C_{x,t}}{|\xi_n|^{p(x)}} - \frac{\alpha_2}{|\xi_n|^{p(x)}}\right) = \infty,
\]

which is absurd since \(D_n(x,t) \to 0\) in \(L^1(Q)\). Let \(\xi^*\) an accumulation point of \((\xi_n)_n\), we have \(|\xi^*| < \infty\) and by continuity of \(a(\cdot, \cdots, \cdot)\), we obtain

\[
a(x,t, u(x,t), \xi^*) - a(x,t, u(x,t), \xi) \cdot (\xi_n - \xi) = 0,
\]

thanks to \((3.3)\), we have \(\xi^* = \xi\), the uniqueness of the accumulation point implies that \(\nabla u_n(x,t) \to \nabla u(x,t)\) a.e. in \(Q\). Since the sequence \(a(x,t, u, \nabla u_n)\) is bounded in \((L^{p(x)}(Q, \omega^*))^N\) and \(a(x,t, u, \nabla u_n) \to a(x,t, u, \nabla u)\) a.e. in \(Q\), Lemma \(2.7\) implies

\[
a(x,t, u_n, \nabla u_n) \to a(x,t, u, \nabla u) \text{ in } (L^{p(x)}(Q, \omega^*))^N.
\]

Let us taking \(\bar{y}_n = a(x,t, u_n, \nabla u_n)\nabla u_n\) and \(\bar{y} = a(x,t, u, \nabla u)\nabla u\), then \(\bar{y}_n \to \bar{y}\) in \(L^1(Q)\), according to the condition \((3.5)\), we have

\[
\alpha_1 |\nabla u_n|^{p(x)} \leq a(x,t, u_n, \nabla u_n)\nabla u_n.
\]

Let \(z_n = |\nabla u_n|^{p(x)}\), \(z = |\nabla u|^{p(x)}\) and \(y_n = \frac{\bar{y}_n}{a}\), \(y = \frac{\bar{y}}{a}\). Then, by Fatou’s Lemma, we obtain

\[
\int_Q 2ydxdt \leq \liminf_{n \to \infty} \int_Q (y_n + y - |z_n - z|)dxdt,
\]

i.e., \(0 \leq \limsup_{n \to \infty} \int_Q |z_n - z|dxdt\)

\[
n \leq \liminf_{n \to \infty} \int_Q |z_n - z|dx \leq \limsup_{n \to \infty} \int_Q |z_n - z|dx \leq 0,
\]

this implies

\[
\nabla u_n \to \nabla u \text{ in } (L^{p(x)}(Q, \omega))^N,
\]

we deduce that

\[
u_n \to u \text{ in } L^{p(x)}(0,T;W_0^{1,p(x)}(\Omega, \omega)),
\]

which completes our proof.

Let \(X = L^{p^-}(0,T;W_0^{1,p(x)}(\Omega, \omega))\), the dual space of \(X\) is \(X^* = L^{p^-}(0,T;W_0^{1,p(x)}(\Omega, \omega))^*\).

**Lemma 2.11.** (See [17] )

\[
W := \left\{ u \in V; u_t \in V^* + L^1(Q) \right\} \hookrightarrow C([0,T];L^1(\Omega))
\]

and

\[
W \cap L^\infty(0,T) \hookrightarrow C([0,T];L^2(\Omega)).
\]

**Definition 2.12.** A monotone map \(T : D(T) \to X^*\) is called maximal monotone if its graph

\[
G(T) = \left\{ (u, T(u)) \in X \times X^* \text{ for all } u \in D(T) \right\},
\]

is not a proper subset of any monotone set in \(X \times X^*\).

Let us consider the operator \(\frac{\partial}{\partial t}\) which induces a linear map \(L\) from the subset

\[
D(L) = \left\{ v \in X : v' \in X^*, v(0) = 0 \right\} \text{ of } X \text{ in to } X^* \text{ by}
\]

\[
\langle Lu, v \rangle_X = \int_0^T \langle v'(t), v(t) \rangle_X dt \quad u \in D(L), \ v \in X.
\]

**Definition 2.13.** A mapping \(S\) is called pseudo-monotone with \(u_n \to u\) and \(L_{u_n} \to Lu\) and \(\lim_{n \to \infty} \sup (S(u_n), u_n - u) \leq 0\), that we have

\[
\lim_{n \to \infty} \sup \langle S(u_n), u_n - u \rangle = 0 \quad \text{and} \quad S(u_n) \to S(u) \quad \text{as} \quad n \to \infty.
\]
3. Assumption and Main Results

Throughout the paper, we assume that the following assumption hold true.

**Assumption (H1)**

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N (N \geq 1) \), \( p \in C_+ (\overline{\Omega}) \) and \( b : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that for every \( x \in \Omega \), \( b(x, \cdot) \) is a strictly increasing \( C^1 \) function with

\[
b(x, 0) = 0. \tag{3.1}
\]

Next, for any \( k > 0 \), there exist \( \lambda_k > 0 \) and functions \( A_k \in L^\infty(\Omega) \) and \( B_k \in L^{p(\cdot)}(\Omega) \) such that

\[
\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left| D_s \left( \frac{\partial b(x, s)}{\partial s} \right) \right| \leq B_k(x), \tag{3.2}
\]

for almost every \( x \in \Omega \) and every \( s \) such that \( |s| \leq k \), we denote by \( D_s (\partial b(x, s) \setminus \partial s) \) the gradient of \( \partial b(x, s) \setminus \partial s \) defined in the sense of distributions.

**Assumption (H2)**

We consider a Leray -Lions operator defined by the formula:

\[
Au = -\text{div} a(x, t, u, \nabla u),
\]

where \( a : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function i.e., (measurable with respect to \( x \) in \( \Omega \) for every \( (s, \xi) \) in \( \mathbb{R} \times \mathbb{R}^N \) and continuous with respect to \( (s, \xi) \) in \( \mathbb{R} \times \mathbb{R}^N \), for almost every \( x \) in \( \Omega \) which satisfies the following conditions there exist \( k \in L^{p'(\cdot)}(Q) \) and \( \alpha > 0, \beta > 0 \) such that for almost every \( (x, t) \in Q \) all \( (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \),

\[
|a(x, t, s, \xi)| \leq \beta \omega^{1/p(x)}(x)|k(x, t)| + \omega^{1/p'(x)}(x)|s|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p'(x)-1}, \tag{3.3}
\]

\[
[a(x, t, s, \xi) - a(x, t, s, \eta)] \cdot (\xi - \eta) > 0 \quad \forall \xi \neq \eta \in \mathbb{R}^N, \tag{3.4}
\]

\[
a(x, t, s, \xi) : \xi \geq \alpha \omega|\xi|^{p(x)}. \tag{3.5}
\]

**Assumption (H3)**

Let \( H : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is a Carathéodory function such that for a.e. \( (x, t) \in Q \) and for all \( s \in \mathbb{R} \), \( \xi \in \mathbb{R}^N \), the growth condition

\[
|H(x, t, s, \xi)| \leq \gamma(x, t) + g(s)\omega|\xi|^{p(x)} \tag{3.6}
\]

is satisfied, where \( g : \mathbb{R} \to \mathbb{R}^+ \) is a bounded continuous positive function that belongs to \( L^1(\mathbb{R}) \), while \( \gamma \in L^1(Q) \).

We recall that, for \( k > 0 \) and \( s \in \mathbb{R} \), the truncation function \( T_k(.) \) defined by

\[
T_k(s) = \begin{cases} 
\frac{s}{k} & \text{if } |s| \leq k \\
\frac{s}{|s|} & \text{if } |s| > k.
\end{cases}
\]

**Definition 3.1.** Let \( f \in L^1(Q) \) and \( b(\cdot, u_0) \in L^1(\Omega) \). A real-valued function \( u \) defined on \( Q \) is renormalized solutions of problem (P) if:

\[
T_k(u) \in L^{p(\cdot)}(0, T; W^{1,p(\cdot)}_0(\Omega, \omega)) \quad \text{for all } k \geq 0, b(x, u) \in L^\infty(0, T; L^{p(\cdot)}(\Omega)), \tag{3.7}
\]

\[
\int_{m \leq |u| \leq m+1} a(x, t, u, \nabla u) \nabla u \, dx \, dt \to 0 \quad \text{as } m \to \infty, \tag{3.8}
\]

\[
\frac{\partial B_s(x, u)}{\partial t} - \text{div} \left( S'(u)a(x, t, u, \nabla u) \right) + S''(u)a(x, t, u, \nabla u) = fS'(u) \quad \text{in } D'(Q), \tag{3.9}
\]

for all \( S \in W^{2,\infty}(\mathbb{R}) \), which are piecewise \( C^1 \) and such that \( S' \) has a compact support in \( \mathbb{R} \), where \( B_S(x, z) = \int_0^z \frac{\partial b(x, r)}{\partial r} S'(r) \, dr \) and

\[
B_S(x, u) \big|_{t=0} = B_S(x, u_0) \quad \text{in } \Omega. \tag{3.10}
\]
Remark 3.2. Equation \((3.9)\) is formally obtained through pointwise multiplication of problem \((\mathcal{P})\) by \(S'(u)\). However, while \(a(x, t, u, \nabla u)\) and \(H(x, t, u, \nabla u)\) do not in general make sense in \((\mathcal{P})\), all the terms in \((3.9)\) have a meaning in \(D'(Q)\). Indeed, if \(M\) is such that \(\text{supp } S \subset [-M, M]\), the following identifications are made in \((3.9)\):

- \(S(u)\) belongs to \(V \cap L^\infty(Q)\). Since \(S\) is a bounded function.
- \(S'(u) a(x, t, u, \nabla u)\) identifies with \(S'(u) a(x, t, T_M(u), \nabla T_M(u))\) a.e. in \(Q\), for any \(\varphi \in D(Q)\), using Hölder inequality

\[
\int_Q S'(u) a(x, t, u, \nabla u) \nabla \varphi dx dt = \int_Q S'(u) a(x, t, T_M(u), \nabla T_M(u)) \nabla \varphi dx dt
\]

\[
\leq C_M \|S'\|_{L^\infty(Q)} \max \left\{ \left( \int_Q |\nabla T_M(u)|^{p(x)} \omega \right)^{\frac{1}{p'}} \left( \int_Q |\nabla T_M(u)|^{p(x)} \omega \right)^{\frac{1}{p'}} \right\} \|\varphi\|_{L^{p'}(Q)},
\]

where \(M > 0\) is such that \(\text{supp } S' \subset [-M, M]\). As \(D(Q)\) is dense in \(V\), we deduce that

\[
\text{div}(S'(u) a(x, t, u, \nabla u)) + \nabla \omega \in V'.
\]

- \(S''(u) a(x, t, u, \nabla u)\) identifies with \(S''(u) a(x, u, T_M(u), \nabla T_M(u)) \nabla T_M(u)\) and \(S''(u) a(x, u, T_M(u), \nabla T_M(u)) \nabla T_M(u) \in L^1(Q)\).
- \(S'(u) H(x, t, u, \nabla u)\) identifies with \(S'(u) H(x, t, T_M(u), \nabla T_M(u))\) a.e. in \(Q\). Since \(T_M(y) \leq M\) a.e. in \(Q\) and \(S'(u) \in L^\infty(Q)\), we see from \((3.6)\) and \((3.7)\) that \(S'(u) H(x, t, T_M(u), \nabla T_M(u)) \in L^1(Q)\).
- \(S'(u) f\) belongs to \(L^1(Q)\).

The above considerations show that equation \((3.9)\) hold in \(D'(Q)\) and that

\[
\frac{\partial B_S(x, u)}{\partial t} \in V^* + L^1(Q).
\]

Due to the properties of \(S\) and \((3.9)\), \(\frac{\partial S(u)}{\partial u} \in V^* + L^1(Q)\), using Lemma 2.11 which implies that \(S(u) \in C^0([0, T); L^1(\Omega))\). So that the initial condition \((3.10)\) makes sense since, due to the properties of \(S\) (increasing) and \((3.2)\), we have

\[
\left| (B_S(x, r) - B_S(x, r')) \right| \leq A_k(x) \left| S(r) - S(r') \right| \text{ for all } r, r' \in \mathbb{R}. \quad (3.11)
\]

Theorem 3.3. Let \(f \in L^1(Q)\), \(p(\cdot) \in C_+ (\overline{\Omega})\) and assume that \(u_0\) is a measurable function such that \(b(\cdot, u_0) \in L^1(\Omega)\). Assume that \((H1) - (H3)\) hold true. Then there, exists a renormalized solution \(u\) of problem \((\mathcal{P})\) in the sense of Definition \((3.1)\).

4. Proof of Main Results.

4.1. Approximate problem. For \(n > 0\), we define approximations of \(b, H, f\) and \(u_0\). First set

\[
b_n(x, r) = b(x, T_n(r)) + \frac{1}{n} r. \quad (4.1)
\]

\(b_n\) is a Carathéodory function and satisfies \((3.2)\). There exist \(\lambda_n > 0\) and functions \(A_n \in L^\infty(\Omega)\) and \(B_n \in L^{p(\cdot)}(\Omega)\) such that

\[
\lambda_n \leq \frac{\partial b_n(x, s)}{\partial s} \leq A_n(x) \text{ and } \left| D_s \left( \frac{\partial b_n(x, s)}{\partial s} \right) \right| \leq B_n(x) \text{ a.e. in } \Omega, s \in \mathbb{R}.
\]

Next, set

\[
H_n(x, t, s, \xi) = \frac{H(x, t, s, \xi)}{1 + \frac{n}{\lambda_n} |H(x, t, s, \xi)|}.
\]

Note that \(|H_n(x, t, s, \xi)| \leq |H(x, t, s, \xi)|\) and \(|H_n(x, t, s, \xi)| \leq n\) for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^N\).
and select \( f_n, u_{0n}, \) and \( b_n \). So that
\[
\begin{align*}
  f_n &\in L^{p'}(Q) \text{ and } f_n \to f \text{ a.e. in } Q, \text{ strongly in } L^1(Q) \text{ as } n \to \infty, \\
u_{0n} &\in D(\Omega), \quad \|b_n(x, u_{0n})\|_{L^1(\Omega)} \leq \|b_n(x, u_n)\|_{L^1(\Omega)}, \\
b_n(x, u_{0n}) &\to b(x, u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega).
\end{align*}
\]

Let us now consider the approximate problem
\[
(P_n) \quad \begin{cases}
  \partial_t u_n + \nabla(u(x, t, u_n, \nabla u_n)) + H_n(x, t, u_n, \nabla u_n) = f_n \quad \text{in } D'(Q), \\
u_0 = 0 \quad \text{on } \partial \Omega \times (0, T) \quad u_n \in L^{p^-}(0, T; W^{1,p(\cdot)}\Omega, \omega)).
\end{cases}
\]

**Theorem 4.1.** Let \( f_n \in L^{p^-}(0, T; W^{1,p(\cdot)}(\Omega, \omega^*)) \), \( p(\cdot) \in C_+(\Omega) \) for fixed \( n \), the approximate problem \((P_n)\) has at least one weak solution \( u_n \in L^{p^-}(0, T; W^{1,p(\cdot)}\Omega, \omega))

**Proof.**
We define the operator \( L_n : L^{p^-}(0, T; W^{1,p(\cdot)}\Omega, \omega) \to L^{p^-}(0, T; W^{1,p(\cdot)}(\Omega, \omega^*)) \) by
\[
\langle L_n u, v \rangle = \int_Q \frac{\partial b_n(x, u)}{\partial t} v dx dt = \int_Q \frac{\partial b_n(x, u_n)}{\partial t} v dx dt \quad \forall u, v \in L^{p^-}(0, T; W^{1,p(\cdot)}\Omega, \omega),
\]
then,
\[
\begin{align*}
  \left| \langle L_n u, v \rangle \right| &\leq \left| \int_0^T \int_\Omega A_n(x) \frac{\partial u}{\partial t} v dx dt \right| = \left| \int_0^T \int_\Omega A_n(x) \frac{\partial u}{\partial \omega} \frac{1}{\omega} \frac{1}{\omega} dx dt \right| \\
&\leq \left( \frac{1}{p^-} + \frac{1}{p^+} \right) \|A_n\|_{L^\infty} \int_0^T \left\| \frac{\partial u}{\partial \omega} \right\|_{L^{p'(\cdot)}(\Omega, \omega^*)} \|v\|_{L^{p(\cdot)}(\Omega, \omega)} dt \\
&\leq C \left( \frac{1}{p^-} + \frac{1}{p^+} \right) \|A_n\|_{L^\infty} \int_0^T \left\| \frac{\partial u}{\partial \omega} \right\|_{L^{p'(\cdot)}(\Omega, \omega^*)} \|v\|_{L^{p(\cdot)}(\Omega, \omega)} dt \\
&\leq C \left( \frac{1}{p^-} + \frac{1}{p^+} \right) \|A_n\|_{L^\infty} \left( \int_0^T \left\| \frac{\partial u}{\partial \omega} \right\|_{L^{p'(\cdot)}(\Omega, \omega^*)} \|v\|_{L^{p(\cdot)}(\Omega, \omega)} dt \right)^\theta \\
&\leq C_2 \|v\|_{L^{p^-}(0, T; W^{1,p(\cdot)}\Omega, \omega))}^\theta 
\end{align*}
\]

with \( \theta = \begin{cases} 
1/p^- & \text{if } \|H_n(x, t, u_n, \nabla u)\|_{L^1(\Omega)} > 1, \\
1/p^+ & \text{if } \|H_n(x, t, u_n)\|_{L^1(\Omega)} \leq 1.
\end{cases} \)

**Lemma 4.2.** Let \( B_n : L^{p^-}(0, T; W^{1,p(\cdot)}\Omega, \omega) \to L^{p^-}(0, T, W^{1,p(\cdot)}(\Omega, \omega^*)) \).

The operator \( B_n = A + G_n \) is

\begin{itemize}
  \item [a)coercive]
\end{itemize}
b) pseudo-monotone

c) bounded and demi continuous.

**Proof. a)** For the coercivity, we have for any $u \in L^p(0, T; W_0^{1, p}(\Omega, \omega))$

\[
\langle B_n u, u \rangle = \langle G_n u, u \rangle + \langle A u, u \rangle
\]

\[
\Rightarrow \langle B_n u, u \rangle - \langle G_n u, u \rangle = \langle A u, u \rangle
\]

then, \[
\langle B_n u, u \rangle - \langle G_n u, u \rangle = \int_Q a(x, t, u, \nabla u) \nabla u \, dxdt
\]

\[
= \int_0^T \int_\Omega a(x, t, u, \nabla u) \nabla u \, dxdt
\]

\[
\geq \int_0^T \alpha \int_\Omega |\nabla u|^p \omega(x) \, dxdt
\]

\[
\geq \alpha \|\nabla u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p \geq \beta \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p
\]

which is due to Poincaré inequality with

\[
\delta = \begin{cases} 
p^- & \text{if } \|\nabla u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))} > 1 \\
p^+ & \text{if } \|\nabla u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))} \leq 1,
\end{cases}
\]

hence, \[
\langle B_n u, u \rangle - \langle G_n u, u \rangle \geq \beta \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p
\]

then, \[
\langle B_n u, u \rangle \geq \beta \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p - C_2 \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p
\]

then, we have

\[
\frac{\langle B_n u, u \rangle}{\|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p} \geq \beta \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p - C_2 \rightarrow +\infty
\]

\[
\Rightarrow \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p \rightarrow +\infty \text{ as } \|u\|_{L^p(0, T; W_0^{1, p}(\Omega, \omega))}^p \rightarrow +\infty
\]

then, $B_n$ is coercive.

**b)** It remains to show that $B_n$ is pseudo-monotone.

Let $(u_k)_k$ a sequence in $L^p(0, T; W_0^{1, p}(\Omega, \omega))$ such that

\[
u_k \rightharpoonup u \text{ in } L^p(0, T; W_0^{1, p}(\Omega, \omega))
\]

\[
L_n u_k \rightharpoonup L_n u \text{ in } L^p(0, T; W_0^{1, p}(\Omega, \omega))
\]

\[
\lim_{k \to \infty} \sup_k \langle B_n u_k, u_k - u \rangle \leq 0
\]

that, we have prove that

\[
B_n u_k \rightharpoonup B_n u \text{ in } L^p(0, T; W_0^{1, p}(\Omega, \omega)) \text{ and } (B_n u_k, u_k) \to (B_n u, u).
\]

By the definition of the operator $L_n$ defined in definition 2.12 we obtain that $u_k$ is bounded in $W_0^{1, p}(\Omega, \omega)$ and since $W_0^{1, p}(\Omega, \omega) \hookrightarrow L^p(\Omega)$, then $u_k \to u$ in $L^p(0, T; W_0^{1, p}(\Omega, \omega))$, then the growth condition \[3.3\] $(\alpha(x, t, u, \nabla u))_k$ is bounded in $(L^p(\Omega, \omega^*)^N$ therefore, there exists a function $\varphi \in (L^p(\Omega, \omega^*)^N$ such that

\[
a(x, t, u_k, \nabla u_k) \to \varphi \text{ as } k \to +\infty.
\]
Similarly, using condition (3.6), \( \left( H_n(x, t, u_k, \nabla u_k) \right)_k \) is bounded in \( L^1(Q) \), then there exists a function \( \psi_n \in L^1(Q) \) such that:

\[
H_n(x, t, u_k, \nabla u_k) \rightarrow \psi_n \quad \text{in} \quad L^1(Q) \quad \text{as} \quad k \rightarrow +\infty.
\] (4.9)

Using (4.7) and (4.10), we obtain

\[
\lim_{k \rightarrow \infty} \sup \langle B_n u_k, u_k \rangle = \lim_{k \rightarrow \infty} \left[ \left( G_n u_k, u_k \right) + \left( A u_k, u_k \right) \right] = \lim_{k \rightarrow \infty} \left[ \int_Q a(x, t, u_k, \nabla u_k) \nabla u_k dx dt + \int_Q H(x, t, u_k, \nabla u_k) u_k dx dt \right] = \int_Q \varphi \nabla u dx dt + \int_Q \psi_n u dx dt
\] (4.10)

thanks to (4.9), we have:

\[
\int_Q H_n(x, t, u_k, \nabla u_k) dx dt \rightarrow \int_Q \psi_n dx dt.
\] (4.12)

therefore,

\[
\lim_{k \rightarrow \infty} \sup \int_Q a(x, t, u_k, \nabla u_k) \nabla u_k \leq \int_Q \varphi \nabla u dx dt
\] (4.13)

on the other hand, using (3.4), we have

\[
\int_Q \left[ a(x, t, u_k, \nabla u_k) - a(x, t, u_k, \nabla u) \right] (\nabla u_k - \nabla u) dx dt \geq 0.
\] (4.14)

Then,

\[
\int_Q a(x, t, u_k, \nabla u_k) \nabla u_k dx dt \geq - \int_Q a(x, t, u_k, \nabla u) \nabla u dx dt + \int_Q a(x, t, u_k, \nabla u) \nabla u_k dx dt + \int_Q a(x, t, u_k, \nabla u) \nabla u_k dx dt
\]

and by (4.8), we get

\[
\lim_{k \rightarrow \infty} \inf \int_Q a(x, t, u_k, \nabla u_k) \nabla u_k dx dt \geq \int_Q \varphi \nabla u dx dt,
\]

this implies, thanks to (4.13) that

\[
\lim_{k \rightarrow \infty} \int_Q a(x, t, u_k, \nabla u_k) \nabla u_k dx dt = \int_Q \varphi \nabla u dx dt.
\] (4.15)

Now, by (4.15), we can obtain

\[
\lim_{k \rightarrow \infty} \int_Q a(x, t, u_k, \nabla u_k) - a(x, t, u_k, \nabla u) \left( \nabla u_k - \nabla u \right) dx dt = 0.
\]
In view of the Lemma \[2.10\] we obtain

\[ u_k \to u \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega)), \]
\[ \nabla u_k \to \nabla u \quad \text{a.e. in} \quad Q. \]

Then,

\[ a(x, t, u_k, \nabla u_k) \to a(x, t, u, \nabla u) \quad \text{in} \quad (L^{p^+}(Q, \omega^+))^N, \]
\[ H_n(x, t, u_k, \nabla u_k) \to H_n(x, t, u, \nabla u) \quad \text{in} \quad L^1(Q), \]

we deduce that

\[ Au_k \to Au \quad \text{in} \quad (L^{p^-}(Q, \omega^+))^N \]

and

\[ G_n u_k \to G_n u \quad \text{in} \quad L^1(Q), \]

which implies

\[ B_n u_k \to B_n u \quad \text{in} \quad L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega)) \]

and

\[ \langle B_n u_k, u_k \rangle \to \langle B_n u, u \rangle \]

completing the proof of assertion (b).

c) Using Hölder’s inequality and the growth condition \[3.3\], we can show that the operator $A$ is bounded, and by using \[4.6\], we conclude that $B_n$ is bounded. To show that $B_n$ is demicontinuous.

Let $u_k \to u$ in $L^p(0,T;W_0^{1,p}(\Omega,\omega))$ and prove that:

\[ \langle B_n u_k, \psi \rangle \to \langle B_n u, \psi \rangle \quad \text{for all} \quad \psi \in L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega)). \]

Since $a(x, t, u_k, \nabla u_k) \to a(x, t, u, \nabla u)$ as $k \to \infty$ a.e. in $Q$. Then, by the growth condition \[3.3\] and Lemma \[2.7\]

\[ a(x, t, u_k, \nabla u_k) \to a(x, t, u, \nabla u) \quad \text{in} \quad (L^{p^+}(Q, \omega^+))^N \]

and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega))$, \( \langle A u_k, \varphi \rangle \to \langle A u, \varphi \rangle \) as $k \to \infty$ similarly, $G_n u_k \to G_n u$ as $k \to \infty$ a.e. in $Q$, then by the \[3.6\] and Lemma \[2.7\] $G_n u_k \to G_n u$ in $L^{p^+}(Q, \omega^+)$ and for all $\varphi \in L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega))$,

\[ \langle G_n u_k, \varphi \rangle \to \langle G_n u, \varphi \rangle \quad \text{as} \quad k \to \infty \]

which implies $B_n$ is demicontinuous.

In view of Theorem \[4.1\], there exists at least one weak solution $u_n \in L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega))$ of the problem $(P_n)$ (See \[12\].)

4.2. A Priori Estimates.

**Proposition 4.3.** Let $u_n$ a solution of the approximate problem $(P_n)$. Then, there exists a constant $C(\text{which does not depend on the } n \text{ and } k)$ such that

\[ ||T_k(u_n)||_{L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega))} \leq kC \quad \forall k > 0. \]

**Proof.**

Let $\varphi \in L^{p^-}(0,T;W_0^{1,p}(\Omega,\omega)) \cap L^{\infty}(Q)$, with $\varphi > 0$. Choosing $v = \exp(G(u_n))\varphi$ as a test function in $(P_n)$, where

\[ G(s) = \int_0^s \frac{g(r)}{\alpha}dr, \]

(the function $g$ appears in \[3.6\]), we have

\[ \int_Q \frac{\partial G(u_n)}{\partial t} \exp(G(u_n))\varphi dx dt + \int_Q a(x, t, u_n, \nabla u_n)\nabla (\exp(G(u_n))\varphi) dx dt \]
\[ + \int_Q H_n(x, t, u_n, \nabla u_n) \exp(G(u_n))\varphi dx dt = \int_Q f_n \exp(G(u_n))\varphi dx dt. \]
In view of (4.16), we obtain
\[\int_Q \frac{\partial b_n(x, u_n)}{\partial t} \exp(G(u_n)) \varphi dx dt + \int_Q a(x, t, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n)) \varphi dx dt\]
\[+ \int_Q a(x, t, u_n, \nabla u_n) \exp(G(u_n)) \nabla \varphi dx dt \leq \int_Q \gamma(x, t) \exp(G(u_n)) \varphi dx dt\]
\[+ \int_Q f_n \exp(G(u_n)) \varphi dx dt + \int_Q g(u_n) |\nabla u_n|^{p(x)} \omega(x) \exp(G(u_n)) \varphi dx dt.\]

By using (3.5), we obtain
\[\int_Q \frac{\partial b_n(x, u_n)}{\partial t} \exp(G(u_n)) \varphi dx dt + \int_Q a(x, t, u_n, \nabla u_n) \exp(G(u_n)) \nabla \varphi dx dt\]
\[\leq \int_Q \gamma(x, t) \exp(G(u_n)) \varphi dx dt + \int_Q f_n \exp(G(u_n)) \varphi dx dt\]
\[\quad (4.16)\]
for all \(\varphi \in L^p(0, T; W_0^{1, p(\cdot)}(\Omega, \omega)) \cap L^\infty(Q), \text{ with } \varphi > 0.\)

On the other hand, taking \(v = \exp(-G(u_n)) \varphi\) as a test function in \((P_n),\) we deduce as in (4.16) that
\[\int_Q \frac{\partial b_n(x, u_n)}{\partial t} \exp(-G(u_n)) \varphi dx dt + \int_Q a(x, t, u_n, \nabla u_n) \exp(-G(u_n)) \nabla \varphi dx dt\]
\[+ \int_Q \gamma(x, t) \exp(-G(u_n)) \varphi dx dt \geq \int_Q f_n \exp(-G(u_n)) \varphi dx dt\]
\[\quad (4.17)\]
for all \(\varphi \in L^p(0, T; W_0^{1, p(\cdot)}(\Omega, \omega)) \cap L^\infty(Q), \text{ with } \varphi > 0.\)

Letting \(\varphi = T_k(u_n)^+ \chi_{(0, r)}\) for every \(r \in [0, T],\) in (4.16), we have
\[\int_0^T \int_\Omega B_{k,G}(x, u_n(t)) dt + \int_\Omega a(x, t, u_n, \nabla u_n) \exp(G(u_n)) \nabla T_k(u_n)^+ dx dt\]
\[\leq \int_\Omega \gamma(x, t) \exp(G(u_n)) T_k(u_n)^+ dx dt + \int_\Omega f_n \exp(G(u_n)) T_k(u_n)^+ dx dt\]
\[+ \int_\Omega B_{k,G}(x, u_n(t)) dt, \quad (4.18)\]
where,
\[B_{k,G}(x, r) = \int_0^r \frac{\partial b_n(x, s)}{\partial s} T_k(s)^+ \exp(G(s)) ds.\]

Due to the definition of \(B_{k,G}^n\) and \(|G(u_n)| \leq \exp\left(\frac{|g|_{L^1(\Omega)}}{\alpha}\right),\) we have
\[0 \leq \int_\Omega B_{k,G}^n(x, u_n(t)) dt \leq k \exp\left(\frac{|g|_{L^1(\Omega)}}{\alpha}\right) \|b(., u_0)\|_{L^1(\Omega)}.\]
\[\quad (4.19)\]
Using (4.19), \(B_{k,G}^n(x, u_n(t)) \geq 0,\) we obtain
\[\int_\Omega a(x, t, u_n, \nabla T_k(u_n)^+) \exp(G(u_n)) \nabla T_k(u_n)^+ dx dt\]
\[\leq k \exp\left(\frac{|g|_{L^1(\Omega)}}{\alpha}\right) \left[\|f_n\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)} + \|b_n(x, u_n)\|_{L^1(\Omega)}\right].\]

Thanks to (3.5), we have
\[\alpha \int_\Omega |\nabla T_k(u_n)^+|^{p(x)} \omega(x) \exp(G(u_n)) dx dt \leq k \exp\left(\frac{|g|_{L^1(\Omega)}}{\alpha}\right) \left[\|f_n\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)} + \|b_n(x, u_n)\|_{L^1(\Omega)}\right]. \quad (4.20)\]
Let us observe that if we take: \( \varphi = \rho(u_n) = \int_0^\infty g(s)\chi_{(s>0)} \) in (4.16), and use (3.5), we obtain
\[
\int_\Omega B_0^n(x,u_n)\frac{T}{0} dx + \alpha \int_0^\infty |\nabla u_n|^{p(x)}(x)\omega(x)\exp(G(u_n)) \omega(x) g(u_n)\chi_{(u_n>0)} \exp(G(u_n))dxdt
\leq \left( \int_0^\infty g(s)ds \right) \exp \left( \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha} \left[ \|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} \right] \right),
\]
where
\[
B_0^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} \rho(s) \exp(G(s))ds,
\]
which implies, using \( B_0^n(x,r) \geq 0 \), we obtain
\[
\alpha \int_\Omega |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dxdt
\leq \|g\|_{L^\infty} \exp \left( \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha} \left[ \|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b_n(x,u_0)\|_{L^1(\Omega)} \right] \right)
\]
then,
\[
\int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))dxdt \leq C_3.
\]
Similarly, taking \( \varphi = \int_0^\infty g(s)\chi_{(s<0)} \) as a test function in (4.17), we conclude that
\[
\int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))dxdt \leq C_4.
\]
Consequently,
\[
\int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))dxdt \leq C_5. \tag{4.21}
\]
Above, \( C_1, ..., C_5 \) are constants independent of \( n \), we deduce that
\[
\int_\Omega |\nabla T_k(u_n)|^{p(x)}\omega(x)dxdt \leq k C_6. \tag{4.22}
\]
Similarly to (4.22), we take \( \varphi = T_k(u_n)^{-}\chi(0,\tau) \) in (4.17) to deduce that
\[
\int_\Omega |\nabla T_k(u_n)|^{-p(x)}\omega(x)dxdt \leq k C_7. \tag{4.23}
\]
Combining (4.22), (4.23) and Remark 2.3, we conclude that
\[
\int_0^\tau \min \left\{ \|T_k(u_n)\|_{W_0^{1,p(\cdot)}(\Omega,\omega)}^+, \|T_k(u_n)\|_{W_0^{1,p(\cdot)}(\Omega,\omega)}^- \right\} dt \leq \rho(\nabla T_k(u_n)) \leq k C_8.
\]
Where \( C_5, C_7, C_8 \) are constants independent of \( n \), Thus, \( T_k(u_n) \) is bounded in \( L^p(0,T;W_0^{1,p(\cdot)}(\Omega,\omega)) \) independently of \( n \) for any \( k > 0 \). Then, we deduce from (4.18), (4.19) and (4.24) that
\[
\int_\Omega B_0^n(x,u_n(\tau))dx \leq k C. \tag{4.25}
\]
4.3. Almost everywhere convergence of the gradients. Now, we turn to proving the almost everywhere convergence of \( u_n \) and \( b_n(x, u_n) \). Consider a non decreasing function \( g_k \in C^2(\mathbb{R}) \) such that: \( g_k(s) = \begin{cases} s & \text{if } |s| \leq \frac{k}{2} \\ k & \text{if } |s| \geq k. \end{cases} \)

Multiplying the approximate equation by \( g_k'(u_n) \), we get
\[
\frac{\partial B^n_k(x, u_n)}{\partial t} - \text{div}(a(x, t, u_n, \nabla u_n)g_k'(u_n)) + a(x, t, u_n, \nabla u_n)g_k''(u_n)\nabla u_n + H_n(x, t, u_n, \nabla u_n)g_k'(u_n) = f_ng_k'(u_n),
\]
where
\[
B^n_k(x, z) = \int_0^z \frac{\partial b_n(x, s)}{\partial s} g_k(s)ds.
\]

As a consequence of (4.24), we deduce that \( g_k(u_n) \) is bounded in \( L^p(0, T; W^{1, p}_0(\Omega, \omega)) \) and \( \frac{\partial B^n_k(x, u_n)}{\partial t} \) is bounded in \( L^1(Q) + V^* \). Due to the properties of \( g_k \) and (3.2), we conclude that \( \frac{\partial b_n(x, u_n)}{\partial u_n} \) is bounded in \( L^1(Q) + V^* \), which implies that \( g_k(u_n) \) is compact in \( L^1(Q) \).

Due to the choice of \( g_k \), we conclude that for each \( k \), the sequence \( T_k(u_n) \) converges almost everywhere in \( Q \), which implies that \( u_n \) converges almost everywhere to some measurable function \( v \) in \( Q \). Thus by using the same argument as in [7, 8, 9], we can show the following lemma.

**Lemma 4.4.** Let \( u_n \) be a solution of the approximate problem \( (P_n) \) then,
\[
\begin{align*}
|u_n| & \to |u| \text{ a.e. in } Q, \\
|b_n(x, u_n)| & \to |b(x, u)| \text{ a.e. in } Q.
\end{align*}
\]

We can deduce from (4.24) that
\[
T_k(u_n) \to T_k(u) \text{ in } L^p(0, T; W^{1, p}_0(\Omega, \omega))
\]
which implies, by using (3.3), that for all \( k > 0 \) there exists \( \varphi_k \in (L^{p'}(Q, \omega^*))^N \), such that
\[
a(x, t, T_k(u_n), \nabla T_k(u_n)) \to \varphi_k \text{ in } (L^{p'}(Q, \omega^*))^N.
\]

**Remark 4.5.** \( b(\cdot, u) \) it belongs to \( L^\infty(0, T; L^1(\Omega)) \).

**Proof.**

Let \( u_n \) be a solution of the approximate problem \( (P_n) \) passing to lim inf in (4.25) as \( n \to \infty \), we obtain
\[
\frac{1}{k} \int_\Omega B_{k,G}(x, u(\tau))dx \leq C, \text{ for a.e. } \tau \text{ in } [0, \tau].
\]

Due to the definition of \( B_{k,G}(x, s) \) and the fact that \( \frac{1}{k} B_{k,G}(x, s) \) converge pointwise to \( f_{sgn(s)} \exp(G(s)) ds \) as \( k \to \infty \), it follows that \( b(\cdot, u) \) belongs to \( L^\infty(0, T; L^1(\Omega)) \).

**Lemma 4.6.** Let \( u_n \) be a solution of the approximate problem \( (P_n) \). Then,
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{|m| \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla u_n) \nabla u_n dx dt = 0. \tag{4.27}
\]

**Proof.**

Set \( \varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n) \) in (4.16), this function is admissible since \( \varphi \in
Let \( L^p(0, T; W_0^{1,p}((\Omega, \omega))) \) and \( \varphi \geq 0 \). Then, we have
\[
\int_Q \frac{\partial b_n(x, u_n)}{\partial t} \exp(G(u_n))\alpha_m(u_n) \, dx \, dt
\]
\[+ \int_{\{m \leq u_n \leq m+1\}} a(x, t, u_n, \nabla u_n) \exp(G(u_n)) \nabla u_n \, dx \, dt \leq \int_Q \gamma(x, t) \exp(G(u_n)) \alpha_m(u_n) \, dx \, dt + \int_T \int_Q |f_n| \exp(G(u_n)) \alpha_m(u_n) \, dx \, dt.
\]
This gives, by setting
\[
B_{n,C}^m(x, r) = \int_0^r \frac{\partial b_n(x, s)}{\partial s} \exp(G(s))\alpha_m(s) \, ds,
\]
and by Young’s Inequality,
\[
\int \Omega B_{n,C}^m(x, u_n)(T) \, dx + \int_{\{m \leq u_n \leq m+1\}} a(x, t, u_n, \nabla u_n) \exp(G(u_n)) \nabla u_n \, dx \, dt
\]
\[\leq \exp \left( \|g\|_{L^1(\Omega)} \right) \left[ \int_{\{|u_n| > m\}} |\gamma| + |f_n| + \|b_n(x, u_n)\|_{L^1(\Omega)} \right] \, dx \, dt.
\]
Since \( B_{n,C}^m(x, u_n)(T) > 0 \) and use (4.28), we obtain
\[
\alpha \int_{\{m \leq u_n \leq m+1\}} \|\nabla u_n\|_p \exp(G(u_n)) \nabla u_n \, dx \, dt
\]
\[\leq \exp \left( \|g\|_{L^1(\Omega)} \right) \left[ \int_{\{|u_n| > m\}} |\gamma| + |f_n| \, dx \, dt + \|b_n(x, u_n)\|_{L^1(\Omega)} \right].
\] (4.28)
Taking \( \varphi = \rho_m(u_n) = \int_0^r g(s) \chi_{\{|u_n| > m\}} \, ds \) as a test function in (4.16), we obtain
\[
\left[ \int_{\Omega} B_{n,m}(x, u_n) \, dx \right]_0^T + \int_Q a(x, t, u_n, \nabla u_n) \exp(G(u_n)) g(u_n) \nabla u_n \chi_{\{|u_n| > m\}} \, dx \, dt
\]
\[\leq \left( \int_{\Omega} g(s) \chi_{\{|u_n| > m\}} \, ds \right) \exp \left( \|g\|_{L^1(\Omega)} \right) \left[ \|f_n\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)} + \|b_n(x, u_n)\|_{L^1(\Omega)} \right],
\]
where \( B_{n,m}(x, r) = \int_0^r \frac{\partial b_n(x, s)}{\partial s} \rho_m(s) \exp(G(s)) \, ds \) which implies, since \( B_{n,m}(x, r) \geq 0 \), by (3.5) and Young’s inequality
\[
\alpha \int_{\{|u_n| > m\}} \|\nabla u_n\|_p \omega(x) g(u_n) \exp(G(u_n)) \, dx \, dt \leq \left( \int_{\Omega} g(s) \, ds \right) \exp \left( \|g\|_{L^1(\Omega)} \right) \left[ \|f_n\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)} + \|b_n(x, u_n)\|_{L^1(\Omega)} \right].
\] (4.29)
Using (4.29) and the strong convergence of \( f_n \) in \( L^1(\Omega) \) and \( b_n(x, u_n) \) in \( L^1(\Omega) \), \( \gamma \in L^1(\Omega) \), \( g \in L^1(\mathbb{R}) \), by Lebesgue’s theorem, passing to limit in (4.28), we conclude that
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{m \leq u_n \leq m+1\}} a(x, t, u_n, \nabla u_n) \, dx \, dt = 0.
\] (4.30)
On the other hand, taking \( \varphi = T_1(u_n - T_m(u_n)) \) as a test function in (4.17) and reasoning as in the proof (4.30), we deduce that
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{-m+1\} \leq u_n \leq -m} a(x, t, u_n, \nabla u_n) \, dx \, dt = 0.
\] (4.31)
By using (4.30) and (4.31), we have
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{m \leq u_n \leq m+1\}} a(x, t, u_n, \nabla u_n) \, dx \, dt = 0.
\] (4.32)
To this end, we prove the strong convergence of truncation of \( T_h(u_n) \) that we will use the following function of one real variable \( s \), which is defined as where \( m > k \),
\[ h_m(s) = \begin{cases} 
1 & \text{if } |s| \leq m \\
0 & \text{if } |s| > m + 1 \\
m + 1 + |s| & \text{if } m \leq |s| \leq m + 1.
\end{cases} \]

Let \( \psi \in D(\Omega) \) be a sequence which converges strongly to \( u_0 \) in \( L^1(\Omega) \).

Set \( w_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi) \) where \( (T_k(u))_\mu \) is the mollification of \( T_k(u) \) with respect to time. Note that \( w_\mu \) is a smooth function having the following properties:

\[
\frac{\partial w_\mu}{\partial t} = \mu(T_k(u) - w_\mu), \quad w_\mu(0) = T_k(\psi), \quad |w_\mu| \leq k, \quad (4.33)
\]

\[
w_\mu \to T_k(u) \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega,\omega)) \quad \text{as } \mu \to \infty. \quad (4.34)
\]

The very definition of the sequence \( w_\mu \) makes it possible to establish the following lemma.

**Lemma 4.7.** (See [2]) For \( k \geq 0 \), we have

\[
\int (T_k(u_\mu) - w_\mu) \exp(G(u_\mu)) (T_k(u_\mu) - w_\mu) h_m(u_\mu) \, dx \, dt \geq \varepsilon(n,m,\mu,\tau).
\]

**Proposition 4.8.** The subsequence of \( u_n \) solution of problem \( (P_n) \) satisfies for any \( k \geq 0 \) following assertion:

\[
\lim_{n \to \infty} \int_Q \left[ a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)) \right] \cdot \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \, dt = 0.
\]

**Proof.**

For \( m > k \), let \( \varphi = (T_k(u_n) - w_\mu)^+ h_m(u_\mu) \in L^p(0,T;W_0^{1,p}(\Omega,\omega)) \cap L^\infty(\Omega) \) and \( \varphi \geq 0 \). If we take this function in (4.16), we obtain

\[
\int (T_k(u_\mu) - w_\mu) \exp(G(u_\mu)) (T_k(u_\mu) - w_\mu) h_m(u_\mu) \, dx \, dt
\]

\[
+ \int (T_k(u_\mu) - w_\mu) a(x,t,u_\mu,\nabla u_\mu) \nabla (T_k(u_\mu) - w_\mu) h_m(u_\mu) \, dx \, dt
\]

\[
- \int (m \leq u_\mu \leq m+1) \exp(G(u_\mu)) a(x,t,u_\mu,\nabla u_\mu) \nabla u_\mu (T_k(u_\mu) - w_\mu)^+ \, dx \, dt
\]

\[
\leq \int_Q (f_n + \gamma) \exp(G(u_\mu)) (T_k(u_\mu) - w_\mu)^+ h_m(u_\mu) \, dx \, dt \quad (4.35)
\]

Observe that

\[
\left| \int_{m \leq u_\mu \leq m+1} \exp(G(u_\mu)) a(x,t,u_\mu,\nabla u_\mu) \nabla u_\mu (T_k(u_\mu) - w_\mu)^+ \, dx \, dt \right|
\]

\[
\leq 2k \exp \left( \frac{|g|_{L^1[\Omega]}}{a} \right) \int_{m \leq u_\mu \leq m+1} a(x,t,u_\mu,\nabla u_\mu) \nabla u_\mu \, dx \, dt.
\]

Tanks to (4.27) the third and fourth integrals on the right hand side tend to zero as \( n \) and \( m \) tend to infinity and by Lebesgue’s theorem, we deduce that the right hand side converges to zero as \( n, m \) and \( k \) tend to infinity. Since \( (T_k(u_\mu) - w_\mu)^+ h_m(u_\mu) \to (T_k(u) - w_\mu)^+ h_m(u) \) in \( L^\infty(\Omega) \) as \( n \to \infty \) and strongly in \( L^p(0,T;W_0^{1,p}(\Omega,\omega)) \) and \( (T_k(u_\mu) - w_\mu)^+ h_m(u_\mu) \to 0 \) in \( L^\infty(\Omega) \) and strongly in \( L^p(0,T;W_0^{1,p}(\Omega,\omega)) \) as \( \mu \to \infty \), it follows that the first and second integrals on the right-hand side of (4.35) converge to zeros as \( n, m, \mu \to \infty \), using [3] Lemma 4.7 and Lemma 2.10, the proof of Proposition 4.8 is complete. Thanks to the Lemma 2.10, we have

\[
T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega,\omega)), \quad \forall k \quad (4.36)
\]
and \( \nabla u_n \to \nabla u \) a.e. in \( Q \), which implies that
\[
a(x, t, T_k(u_n), \nabla T_k(u_n)) \to a(x, t, T_k(u), \nabla T_k(u)) \quad \text{in} \quad (L^{p'(1)}(Q, \omega^*))^N.
\]
(4.37)

4.4. Equi-Integrability of the non Linearity Sequence.

Proposition 4.9. Let \( u_n \) be a solution of problem \( (P_n) \). Then \( H_n(x, t, u_n, \nabla u_n) \to H(x, t, u, \nabla u) \) strongly in \( L^1(Q) \).

Proof. By using Vitali’s theorem. Since \( H_n(x, t, u_n, \nabla u_n) \to H(x, t, u, \nabla u) \) a.e. in \( Q \), considering now, \( \varphi = \rho_h(u_n) = \int_0^{u_n} g(s) \chi_{\{s > h\}} ds \) as a test function in (4.16), we obtain
\[
\left[ \int_\Omega B^a_n(x, u_n) dx \right]^T_0 + \int_Q a(x, t, u_n, \nabla u_n) \nabla u_n g(u_n) \chi_{\{u_n > h\}} \exp(G(u_n)) dx dt \\
\leq \left( \int_h^\infty g(s) \chi_{\{s > h\}} ds \right) \exp \left( \frac{\|g\|_{L^1(Q)}}{\alpha} \right) \left[ \|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x, u_n)\|_{L^1(Q)} \right],
\]
where \( B^a_n(x, r) = \int_0^r \frac{\partial h_n(x, s)}{\partial s} \rho_h(s) \exp(G(s)) ds \), which implies, in view of \( B^a_n(x, r) \geq 0 \) and (3.5)
\[
\alpha \int_{\{u_n > h\}} |\nabla u_n|^{p(x)} \omega(x) g(u_n) \exp(G(u_n)) dx dt \\
\leq \left( \int_h^\infty g(s) ds \right) \exp \left( \frac{\|g\|_{L^1(Q)}}{\alpha} \right) \left[ \|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \|b_n(x, u_n)\|_{L^1(Q)} \right]
\]
and since \( g \in L^1(\mathbb{R}) \), we deduce that
\[
\lim_{h \to \infty} \sup_{n \in \mathbb{N}} \int_{\{u_n > h\}} |\nabla u_n|^{p(x)} \omega(x) g(u_n) dx dt = 0.
\]
Similarly, taking \( \varphi = \rho_h(u_n) = \int_0^{u_n} g(s) \chi_{\{s < -h\}} ds \) as a test function in (4.17), we conclude that: \( \lim_{h \to \infty} \sup \int_{\{u_n < -h\}} |\nabla u_n|^{p(x)} \omega(x) g(u_n) dx dt = 0 \).
Consequently, \( \lim_{h \to \infty} \sup \int_{\{u_n > h\}} |\nabla u_n|^{p(x)} \omega(x) g(u_n) dx dt = 0 \).

Which implies, for \( h \) large enough and for a subset \( E \) of \( Q \),
\[
\lim_{\text{meas} E \to 0} \int_E |\nabla u_n|^{p(x)} \omega(x) g(u_n) dx dt \leq \|g\|_{\infty} \lim_{\text{meas} E \to 0} \int_E |\nabla T_h u_n|^{p(x)} \omega(x) dx dt \\
+ \int_{\{u_n > h\}} |\nabla u_n|^{p(x)} \omega(x) g(u_n) dx dt,
\]
since \( g(u_n)|\nabla u_n|^{p(x)} \omega(x) \) is equi-integrable. Thus we have shown that
\[
g(u_n)|\nabla u_n|^{p(x)}(x) \omega(x) \to g(u)|\nabla u|^{p(x)}(x) \omega(x) \quad \text{stongly in} \quad L^1(Q).
\]
Consequently, by using (3.6), we conclude that
\[
H_n(x, t, u_n, \nabla u_n) \to H(x, t, u, \nabla u) \quad \text{strongly in} \quad L^1(Q).
\]
(4.38)

4.5. Concluding the proof of Theorem 3.3
a) Proof that $u$ satisfies (3.8). For any fixed $m \geq 0$, we have

$$\int_{\{m \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla u_n) \nabla u_n dxdt$$

$$= \int_Q a(x, t, u_n, \nabla u_n) [\nabla T_{m+1}(u_n) - \nabla T_m(u_n)] dxdt$$

$$= \int_Q a(x, t, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) dxdt$$

$$- \int_Q a(x, t, T_m(u_n), \nabla T_m(u_n)) \nabla T_m(u_n) dxdt.$$  

According to (4.36) and (4.37), one can pass to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\lim_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla u_n) \nabla u_n dxdt$$

$$= \int_Q a(x, t, T_{m+1}(u), \nabla T_{m+1}(u)) \nabla T_{m+1}(u) dxdt$$

$$- \int_Q a(x, t, T_m(u), \nabla T_m(u)) \nabla T_m(u) dxdt$$

$$= \int_{\{m \leq |u| \leq m+1\}} a(x, t, u, \nabla u) \nabla u dxdt. \quad (4.39)$$

Taking the limit as $m \to \infty$ in (4.39) and using the estimate (4.27), shows that $u$ satisfies (3.8).

b) Proof that $u$ satisfies (3.9)

Let $S \in W^{2,\infty}(\mathbb{R})$ be such that $S'$ has a compact support. Let $M > 0$ such that $	ext{supp}(S') \subseteq [-M, M]$. Pointwise multiplication of the approximate problem $(P_n)$ by $S'(u_n)$, leads to

$$\frac{\partial B^n_S(x, u_n)}{\partial t} - \nabla \left[ S'(u_n) a(x, t, u_n, \nabla u_n) \right] + S''(u_n) a(x, t, u_n, \nabla u_n) \nabla u_n$$

$$+ H_n(x, t, u_n, \nabla u_n) S'(u_n) = f_n S'(u_n) \quad \text{in} \ D'(Q). \quad (4.40)$$

In what follows, we pass to the limit in (4.40) as $n \to \infty$.

- Limit of $\frac{\partial B^n_S(x, u_n)}{\partial t}$.

Since $S$ is bounded and continuous, $u_n \to u$ a.e. in $Q$ implies that $B^n_S(x, u_n)$ converge to $B_S(x, u)$ a.e. in $Q$ and $L^\infty$ weakly

$$\text{Then,} \quad \frac{\partial B^n_S(x, u_n)}{\partial t} \to \frac{\partial B_S(x, u)}{\partial t} \quad \text{in} \ D'(Q), \quad \text{as} \ n \to \infty.$$  

- Limit of $-\nabla \left[ S'(u_n) a(x, t, u_n, \nabla u_n) \right]$.

Since $	ext{supp}(S') \subseteq [-M, M]$, we have, for $n \geq M$

$$S'(u_n) a(x, t, u_n, \nabla u_n) = S'(u) a(x, t, T_M(u_n), \nabla T_M(u_n)) \quad \text{a.e. in} \ Q.$$  

The pointwise convergence of $u_n$ to $u$ and (4.37) and the boundedness of $S'$ yied, as $n \to \infty$,

$$S'(u_n) a(x, t, u_n, \nabla u_n) \to S'(u) a(x, t, T_M(u), \nabla T_M(u)) \quad \text{in} \ (L^{p'}(Q, \omega^*))^N \quad (4.41)$$

as $n \to \infty$, $S'(u) a(x, t, T_M(u), \nabla T_M(u))$ has been denoted by $S''(u) a(x, t, u, \nabla u)$ in equation...
As a conclusion, the proof of Theorem 3.3 is complete.

Consider the "energy" term

\[ S''(u_n) \alpha(x,t,u_n,\nabla u_n) \nabla u_n = S''(u_n) \alpha(x,t,T_M(u_n),\nabla T_M(u_n)) \nabla T_M(u_n) \text{ a.e. in } Q. \]

The pointwise convergence of \( S'(u_n) \) to \( S'(u) \) and (4.37) as \( n \to \infty \) and the boundedness of \( S'' \) yield

\[ S''(u_n) \alpha(x,t,u_n,\nabla u_n) \nabla u_n \to S''(u) \alpha(x,t,T_M(u),\nabla T_M(u)) \nabla T_M(u) \text{ in } L^1(Q). \]

Recall that

\[ S''(u) \alpha(x,t,T_M(u),\nabla T_M(u)) \nabla T_M((u)) = S''(u) \alpha(x,t,u,\nabla u) \nabla u \text{ a.e. in } Q. \]

We have

\[ S'(u_n) H_n(x,t,u_n,\nabla u_n) \to S'(u) H(x,t,u,\nabla u) \text{ strongly in } L^1(Q) \text{ as } n \to \infty. \]

As a consequence of the above convergence result, we are in a position to pass to the limit as \( n \to \infty \) in equation (4.40) and to conclude that \( u \) satisfies (3.9).

c) **Proof that \( u \) satisfies (3.10)**

\( S \) is bounded and \( B^2_S(x,u_n) \) is bounded in \( L^\infty(Q) \). Secondly by (4.40), we have \( \partial B^2_S(x,u_n) \) is bounded in \( L^1(Q) + V^* \).

As a consequence, an Aubin type Lemma (see, e.g., [18] implies that \( B^2_S(x,u_n) \) lies in a compact set in \( C^0([0,T],L^1(\Omega)) \).

It follows that on the hand, \( B^2_S(x,u_n) \big|_{t=0} = B^2_S(x,u_0) \) converge to \( B_S(x,u) \big|_{t=0} \) strongly in \( L^1(\Omega) \) implies that: \( B_S(x,u) \big|_{t=0} = B_S(x,u_0) \) in \( \Omega \).

As a conclusion, the proof of Theorem 3.3 is complete.

**References**


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